Ch – 1 Relations and Functions

Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions.

Relation

A Relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product set A × B. i.e. R ⊂ A × B. A relation in a set A is the subset of Cartesian product AXA. If a and b are elements of set A and a is related to b then we write as (a,b) ∈ R or aRb. The set of all first elements in the ordered pair (a,b) in a relation R, is called the domain of the relation R and the set of all second elements called images, is called the range of R.

Types of relations

Empty relation: A relation R in a set A is called *empty relation*, if no element of A is related to any element of A. i.e., R = ϕ ⊂ A × A.

Universal relation: A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e. R = A × A.

Equivalence relation. A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive

A relation R in a set A is called

(i) ***Reflexive***, if *(a, a)* ∈ R, for every *a* ∈ A,

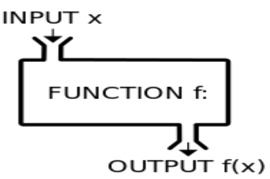
(ii) ***Symmetric***, if *(a, b)* ∈ R implies that *(a, b)* ∈ R, for all *a*, *b* ∈ A.

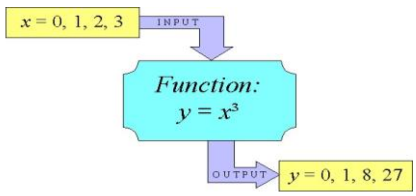
(iii) ***Transitive,*** if *(a, b)*∈ R and *(b, c)* ∈ R implies that *(a, c)* ∈ R *a, b, c* ∈ A.

Function

A relation f from a set A to a set B is said to be function if every element of set A has one and only one image in set B. The notation f : X →Y means that f is a function from X to Y. X is called the domain of f and Y is called the co-domain of f.

Given an element x ∈ X, there is a unique element y in Y that is related to x. The unique element y to which f relates x is denoted by f (x) and is called f of x, or the value of f at x, or the image of x under f. The set of all values of f(x) taken together is called the range of f or image of X under f.
Symbolically, Range of f = {y ∈ Y | y = f (x), for some x in X}.



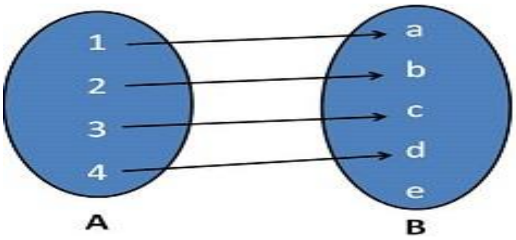


Types of functions:

***One-one* (or *injective*):**

A function *f:* X → Y is defined to be *one-one* (or *injective*), if the images of distinct elements of X under *f* are distinct,
i.e., for every *a, b in* X, *f* (*a*) = *f* (b) implies *a* = *b*. Otherwise, *f* is called *many one*.

To disprove the result, we need to use its contrapositive statement: $a \neq b \Longrightarrow f(a) \neq f(b)$.



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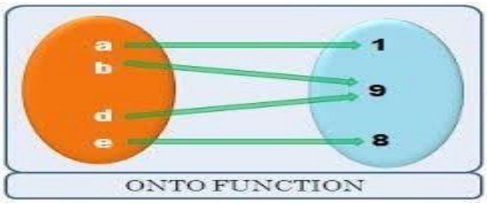
Onto (or surjective):

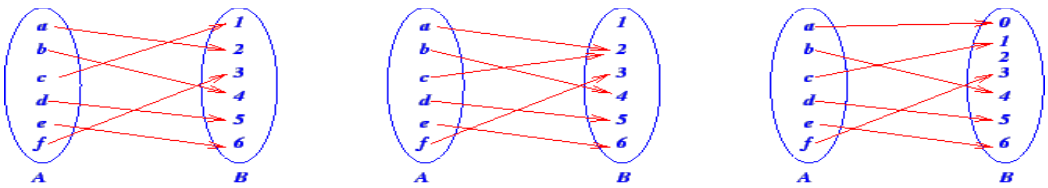
A function f : X →Y is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under *f*, i.e., for every *y in* Y, there exists an element *x* in X such that $f(x) = y$.

OR $Co - domf = Rangef$



Bijective function:

A function f : X→ Y is said to be *bijective* if *f* is both one-one and onto.



(p) a bijection (q) not a bijection (r) not a bijection

Important points on Relation & Functions:

1. *The total number of relations from a set A having 'm' elements to a set B having 'n' elements is* 2^{mn}.
2. *The total number of relations in a set A having 'n' elements is* 2^{n^2}.
3. *The number of reflexive relations in a set having n elements is* 2^{n^2-n}.
4. *The number of symmetric relations in a set having n elements is* $2^{\frac{n^2+n}{2}}$.
5. *The total number of equivalence relations in a set A having 'n' elements:*

   By Bell numbers

n	***1***			
1	*1*	***2***		
2	*2*	*3*	***5***	
3	*5*	*7*	*10*	***15***

4	*15 20 27 37* ***52***
	and so on

6. *The number of injective/one to one function(s) from a set A having 'm' to another set B having 'n' elements with* $n \geq m$ *is* $\frac{m!}{(m-n)!}$.
7. *The number of injective/one to one function(s) from a set A having 'm' to another set B having 'n' elements is* n^m.
8. *The number of onto functions from a set A having 'm' elements to a set B having 'n' elements* $m \geq n$ *is* $2^m - 2$
9. The number of *bijective functions in a set A having 'n' elements is n!*

Ch – 2 Inverse Trigonometric Functions

Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions.

. *The following table gives the inverse trigonometric function (principal value branches) along with their **domains** and **ranges:***

Functions	***Domain***	***Range (Principal Value Branches)***
$y=\sin^{-1}x$	$[-1,\ 1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}x$	$[-1,\ 1]$	$[0,\ \pi]$
$y=\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$y=\cot^{-1}x$	R	$(0,\ \pi)$
$y=\operatorname{cosec}^{-1}x$	$R-(-1,\ 1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y=\sec^{-1}x$	$R-(-1,\ 1)$	$[0,\ \pi]-\left\{\frac{\pi}{2}\right\}$

$\sin^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}x,\ x\in R-(-1,\ 1)$	$\sin^{-1}(-x)=-\sin^{-1}x,\ x\in[-1,\ 1]$
$\cos^{-1}\frac{1}{x}=\sec^{-1}x,\ x\in R-(-1,\ 1)$	$\tan^{-1}(-x)=-\tan^{-1}x,\ x\in R$
$\tan^{-1}\frac{1}{x}=\cot^{-1}x,\ x>0$	$\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x,\ \lvert x\rvert\geq 1$
$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\ x\in[-1,1]$	$\cos^{-1}(-x)=\pi-\cos^{-1}x,\ x\in[-1,\ 1]$
$\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2},\ \lvert x\rvert\geq 1$	$\sec^{-1}(-x)=\pi-\sec^{-1}x,\ \lvert x\rvert\geq 1$
$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ x\in R$	$\cot^{-1}(-x)=\pi-\cot^{-1}x,\ x\in R$

SOME ILUSTRATIONS :

1. Domain of $\cos^{-1}(2x-1)$ is $[0\ ,\ 1]$

As $-1\leq 2x-1\leq 1\Rightarrow 0\leq 2x\leq 2\Rightarrow 0\leq x\leq 1$

2. Principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equal to $\frac{5\pi}{6}$

As $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

3. Principal value of $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ is equal to $\frac{\pi}{12}$

IMPORTANT TRIGONOMETRIC RESULTS & SUBSTITUTIONS

** **<u>Formulae for t-ratios of Allied Angles :</u>**

All T-ratio changes in $\frac{\pi}{2} \pm \theta$ and $\frac{3\pi}{2} \pm \theta$ while remains unchanged in $\pi \pm \theta$ and $2\pi \pm \theta$.

$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta$ $\quad$ $\sin\left(\frac{3\pi}{2} \pm \theta\right) = = \cos\theta$

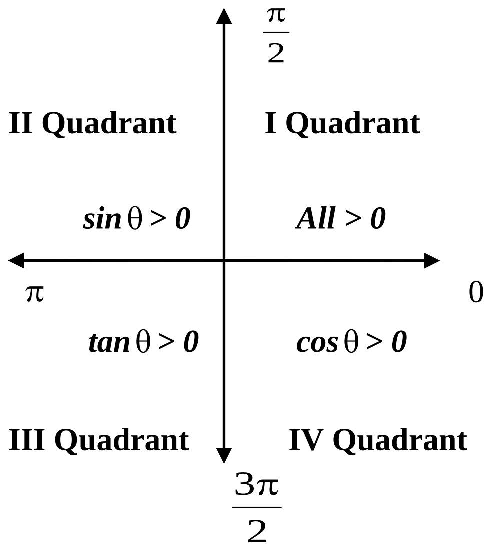
$\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta$ $\quad$ $\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin\theta$

$\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot\theta$ $\quad$ $\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot\theta$

$\sin(\pi \pm \theta) = \mp\sin\theta$ $\quad$ $\sin(2\pi \pm \theta) = \pm\sin\theta$

$\cos(\pi \pm \theta) = = \cos\theta$ $\quad$ $\cos(2\pi \pm \theta) = \cos\theta$

$\tan(\pi \pm \theta) = \pm\tan\theta$ $\quad$ $\tan(2\pi \pm \theta) = \pm\tan\theta$



** **Sum and Difference formulae :**

sin(A + B) = sin A cos B + cos A sin B
sin(A – B) = sin A cos B – cos A sin B
cos(A + B) = cos A cos B – sin A sin B
cos(A – B) = cos A cos B + sin A sin B

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$, $\tan\left(\frac{\pi}{4} + A\right) = \frac{1 + \tan A}{1 - \tan A}$,

$\tan\left(\frac{\pi}{4} - A\right) = \frac{1 - \tan A}{1 + \tan A}$, $\cot(A + B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$ $\quad$ $\cot(A - B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$

$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

$\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

****Formulae for the transformation of a product of two circular functions into algebraic sum of two circular functions and vice-versa.**

2 sinA cos B = sin (A + B) + sin(A – B)
2 cosA sin B = sin (A + B) – sin(A – B)
2 cosA cos B = cos (A + B) + cos(A – B)
2 sinA sin B = cos (A – B) – cos(A + B)

$\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$, $\quad$ $\sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}$.

$\cos C + \cos D = 2\cos\frac{C + D}{2}\cos\frac{C - D}{2}$, $\quad$ $\cos C - \cos D = -2\sin\frac{C + D}{2}\sin\frac{C - D}{2}$.

** **Formulae for t-ratios of múltiple and sub-múltiple angles** :

$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1 + \tan^2 A}$.

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$1 + \cos 2A = 2\cos^2 A$ $\quad$ $1 - \cos 2A = 2\sin^2 A$ $\quad$ $1 + \cos A = 2\cos^2 \frac{A}{2}$ $\quad$ $1 - \cos A = 2\sin^2 \frac{A}{2}$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$, $\quad$ $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$.

$\sin 3A = 3\sin A - 4\sin^3 A$, $\quad$ $\cos 3A = 4\cos^3 A - 3\cos A$

$\sin 15^o = \cos 75^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$. & $\cos 15^o = \sin 75^o = \frac{\sqrt{3}+1}{2\sqrt{2}}$,

$\tan 15^o = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75^o$ & $\tan 75^o = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15^o$.

$\sin 18^o = \frac{\sqrt{5}-1}{4} = \cos 72^o$ and $\cos 36^o = \frac{\sqrt{5}+1}{4} = \sin 54^o$.

$\sin 36^o = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^o$ and $\cos 18^o = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^o$.

$\tan\left(22\frac{1}{2}\right)^o = \sqrt{2} - 1 = \cot 67\frac{1}{2}^o$ and $\tan\left(67\frac{1}{2}\right)^o = \sqrt{2} + 1 = \cot\left(22\frac{1}{2}\right)^o$.

** **Properties of Triangles** : In any Δ ABC,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ [Sine Formula]

$\cos A = \frac{b^2+c^2-a^2}{2bc}$, $\cos B = \frac{c^2+a^2-b^2}{2ca}$, $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

** **Projection Formulae** : $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$

** **Some important trigonometric substitutions :**

$\sqrt{a^2+x^2}$	Put $x = a\tan\theta$ or $a\cot\theta$
$\sqrt{x^2-a^2}$	Put $x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
$\sqrt{a+x}$ or $\sqrt{a-x}$ or both	Put $x = a\cos 2\theta$
$\sqrt{a^n+x^n}$ or $\sqrt{a^n-x^n}$ or both	Put $x^n = a^n\cos 2\theta$
$\sqrt{1+\sin 2\theta}$	$= \sin\theta + \cos\theta$
$\sqrt{1-\sin 2\theta}$	$= \cos\theta - \sin\theta,\ 0 < \theta < \frac{\pi}{4}$
	$= \sin\theta - \cos\theta,\ \frac{\pi}{4} < \theta < \frac{\pi}{2}$

****General solutions:**

*$\cos\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$

*$\sin\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2},\ n \in Z$

*$\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$

*$\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ n \in Z$

*$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in Z$

*$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$

Ch – 3 Matrices

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operations on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

Matrix

A set of m×n numbers (weather real or complex) or functions arranged in the form of a rectangular array of m horizontal lines (called rows) and n vertical lines (called columns) is called a m × n matrix. A matrix is generally denoted by capital letter of English alphabet and an element of a matrix is denoted by a small letter. A matrix with m rows and n columns is called a matrix of order m by n.

Remark : (i) A matrix is an arrangement of numbers or functions and not a value.

(ii) Order of a matrix is written as m × n (read as m by n).

(ii) A general matrix of order m by n is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

i = 1, 2, 3, ……….., m and j = 1, 2, 3, ……….., n

a_{ij} is called the (i, j)th element of the matrix A.

Types of matrices:

1. Row matrix: A matrix having only one row is called a row matrix.

e.g. A = [1 3 5] is a row matrix of order 1× 3.

3. Zero or Null matrix : A matrix is said to be a zero matrix if all its elements are zero.

i.e. A = [a_{ij}] is null matrix if a_{ij} = 0 for all i,j.

Example: $O = [\,0 \quad 0 \quad 0\,]$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. Square matrix: A matrix $A = [a_{ij}]_{mxn}$ is said to be a square matrix if the no. of rows and columns in the matrix are same.

Example: $A = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 5 & 6 \\ 4 & 0 & 8 \end{bmatrix}$

5. Diagonal elements of a square matrix : The elements a_{ij} are called diagonal elements of a square matrix $A = [a_{ij}]_{mxn}$ if i = j. i.e. $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal elements of a square matrix. The line containing the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is called the principal diagonal.

6. Diagonal matrix: A square matrix is said to be a diagonal matrix if its non- diagonal elements are zeros.

e.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is a diagonal matrix.

7. Scalar Matrix:

A square matrix is said to be a scalar matrix if its non- diagonal elements are zeros and all diagonal elements are equal.

As for example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8.Unit matrix : A diagonal matrix is called a unit matrix if all the diagonal elements are unity.

As for example : $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9.Equal Matrices: Two matrices $A = [a_{ij}]_{mx\,n}$ and $B = [b_{ij}]_{mx\,n}$ of the same order are equal if $a_{ij} = b_{ij}$ for all i, j

10. Upper Triangular matrix: A square matrix $A = [a_{ij}]_{nxn}$ is called an upper triangular matrix iff $a_{ij} = 0$ for all $i > j$ i.e. all elements below the principal diagonal are zero.

As for example : $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 2 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

11.Lower Triangular matrix: A square matrix $A = [a_{ij}]_{nxn}$ is called an lower triangular matrix iff $a_{ij} = 0$ for all $i < j$ i.e. all elements above the principal diagonal are zero.

As for example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$

12.Triangular matrix: A square matrix $A = [a_{ij}]_{nxn}$ is called a triangular matrix if it is either upper triangular or lower triangular.

Addition of matrices

Let A and B be two matrices of the same order $m \times n$. Then their sum denoted by A + B is also a matrix of the same order $m \times n$ and is obtained by adding the corresponding elements of A and B.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

As for example :

If $A = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 9 & 0 \end{bmatrix}$, then $A + B = \begin{bmatrix} 13 & 8 \\ 15 & 2 \end{bmatrix}$,

Properties of matrix addition

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are of the same order then

(i) Matrix addition is commutative i.e. A + B = B + A
(ii) Matrix addition is associative i.e. (A + B) + C = A + (B + C)
(iii) Existence of additive Identity : If A is any matrix of order $m \times n$ then there exists a null matrix O of order $m \times n$ such that A + O = O + A = A
(iv) Existence of additive Inverse : A + (-A) = (-A) + A = O
(v) Cancellation Laws: A + B = A + C ⇒B =C and B + A = C + A ⇒ B =C

Properties of scalar multiplication

If A and B are two matrices of the same order and k, l are scalars then

(i) k(A + B) = kA + k B
(ii) (-k)A = -(k A) = k (-A)
(iii) I A = A
(iv) (-1) A = - A

Difference of two matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of the same order $m \times n$. Then their difference denoted by $A - B$ is also a matrix of the same order $m \times n$ and is defined as

$$A - B = [a_{ij} - b_{ij}]_{m \times n}$$

Multiplication of Matrices

Two matrices A and B are said to be conformal for the product if the no. of columns in A is equal to the no. of rows in B.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{m \times n}$ be two matrices then the product of A and B is a matrix $C = [c_{ik}]_{m \times p}$ where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots\ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$$

Properties of matrix multiplication

(i) Matrix multiplication is not commutative i.e. AB ≠ BA
(ii) Matrix multiplication is associative i.e. (AB)C = A(BC)
(iii) Matrix multiplication is distributive over matrix addition i.e. A(B+C) = AB + AC

(iv) If A is an $m \times n$ matrix and I is the identity matrix of order $n \times n$ then I A = A = A I

(v) If A is an $m \times n$ matrix and O is the null matrix of order $n \times p$ then A O = O

(vi) In a matrix multiplication the product of two non-zero matrices may be a zero matrix.

Transpose of a matrix The transpose of a matrix A is denoted by A^T or A' which is obtained by interchanging its rows and columns.

Properties of Transpose of a Matrix

(i) $(A^T)^T = A$ (iii) $(A+B)^T = A^T + B^T$

(ii) $(KA)^T = K A^T$ (iv) $(AB)^T = B^T A^T$

(v) $(ABC)^T = C^T B^T A^T$

Symmetric matrix A square matrix A = $[a_{ij}]_{nxn}$ is said to be a symmetric matrix if transpose of A is equal to matrix A. i.e. if A^T = A.

For example: Let A= $\begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$, then $\begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix}$

∴ A is said is symmetric matrix.

Skew-symmetric matrix A square matrix A = $[a_{ij}]_{nxn}$ is said to be a skew symmetric matrix if $A^T = -$A.

For example:

Let A = $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ = - $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ = -A

∴ A is skew-symmetric matrix.

Note:

(1) All main diagonal elements of a skew symmetric matrix are zero.

(2) Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.

(3) All positive integral powers of a symmetric matrix are symmetric.

(4) odd positive integral powers of a skew-symmetric matrix are skew-symmetric.

Invertible matrices

Let A be a square matrix of order n, If then there exist a square matrix B such that AB = I_n = BA, then B is called the inverse of A and is denoted by A^{-1}.

∴ $AA^{-1} = I = A^{-1}A$

If A and B are invertible square matrices of the same order then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Ch – 4 Determinants

Determinants of a square matrix (up to 3x3 matrices), minors, co-factors, and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency, and number of solutions of system of linear equations by example, solving system of linear equations in two or three variables (having unique solution) using inverse of matrix.

Determinants

Every square matrix can be associated to an expression or a number which is known as its determinant. If A is a square matrix then its determinant is donated by det A or |A|.
Note:
(i) Only square matrices have determinants. The determinant of non-square matrices is not defined.
(ii) A matrix is an arrangement of numbers and hence it has no fixed value while each determinant has a fixed value.
(iii) A determinant having n rows and having n columns is known as a determinant of order n.

Value of a determinant

Determinant of a matrix of order one

Let A = [a_{11}] be a square matrix of order 1 then $|A| = |a_{11}|$ and the value of determinant is the number itself. i.e. $|a_{11}| = a_{11}$

Remark: A determinant of order 1 should not be confused with the absolute value of the number a_{11}.

As for example: (i) If A= $[5]$, then $|5| = 5$

(ii) If A= $[-2]$, then $|-2| = -2$

Determinant of a matrix of order 2

Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2,

then det A = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and its value is $a_{11}.a_{22} - a_{21}.a_{12}$

Note: The numbers $a_{11,}a_{12,}a_{21,}a_{22}$ are called the elements of the determinant.

As for example: (i) If A = $\begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$, then $|A| = 3 \times 6 - 4 \times 2 = 10$

(ii) $\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix} = \cos\theta \, . \cos\theta - \sin\theta \, . (-\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

Determinant of a matrix of order 3

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3,

then det A = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}.a_{32} - a_{22}.a_{31})$

***As for example*:** (i) $\begin{vmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \\ 0 & 4 & 7 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ 4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}$ = 3.18+27+5.4 = 88

(ii) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ = 3(0-5) –(- 1)(0+3) +(-2)(0-0) = - 15 + 3 +0 = - 12

IMPORTANT POINTS TO REMEMBER

1. Only square matrices have determinants.
2. In case of matrices, we take out any common factor from each element of matrix, while in the case of determinants we can take out common factor from any one row or any one column of the determinant.
3. If area is given, then we take both positive and negative values of the determinant for calculation.
4. If we want to prove that three points are collinear, we show that the area formed by these three points is equal to zero.
5. A square matrix of order n is invertible if it is non-singular.
6. If A is an invertible matrix of order n, then $A^{-1} = \frac{1}{|A|}$ adj A where $|A| \neq 0$
7. The inverse of an invertible symmetric matrix is symmetric.
8. If a system of equations has one or more solutions, then it is said to be a consistent system of equations otherwise it is in-consistent.
9. A system of linear equations may or may not be consistent.
10. A consistent system may or may not have a unique solution.
11. Every invertible matrix possesses a unique inverse.
12. If A' is transpose of a square matrix A then $|A'| = |A|$

    As for example: If A = $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, then $|A| = 2.4 - 5.3 = 8 - 15 = -7$

    Now A'= $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$, then $|A'| = 2.4 - 3.5 = 8 - 15 = -7$

13. If A =$[a_{ij}]_{n\times n}$ then | k A | = k^n $|A|$

    As for example:
    If A = $[a_{ij}]_{3\times 3}$, then $|kA|$ = k^3 $|A|$

14. If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$
15. If A is a non-singular matrix of order n, then $|A^{-1}| = \frac{1}{|A|}$
16. If A and B are non-singular matrices of same order, then$|AB|=|A||B|$ and hence $|A.adjA| = |A|^n$

Illustrations

1. If A is square matrix of order 3 with value of its determinant 4, then $|3A| = 3^3\ |A| = 27 \times 4 = 108$
2. If A is a square matrix order 3 such that $|A| = 4$, and $|KA|$= 500, find K.
   Solution: ∵ $|KA|$= 500
   $\Rightarrow k^3|A| = 500$
   $\Rightarrow k^3.4 = 500$
   $\Rightarrow k^3 = 500/4 = 125$
   $\Rightarrow k = 5$

* **Area of a Triangle:** area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

* Equation of a line passing through (x_1, y_1)&$(x_2, y_2)$is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

*__Minor__ of an element $a_{ij}$of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

* Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

* **Cofactor** of an element a_{ij}, denoted by $A_{ij}$is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where $M_{ij}$is minor of a_{ij}.

* If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

*__Adjoint of matrix__ :

If $A = [a_{ij}]$ be a square matrix then transpose of a matrix $[A_{ij}]$, where A_{ij} is the cofactor of A_{ij} element of matrix A, is called the adjoint of A.

Adjoint of A = Adj. A = $[A_{ij}]^T$.

A(Adj.A) = (Adj. A)A = | A| I.

* If A be any given square matrix of order n, then A(adjA) = (adjA) A = |A| I,

* If A is a square matrix of order n, then $|adj(A)| = |A|^{n-1}$.

*__Inverse of a matrix__ :Inverse of a square matrix A exists, if A is non-singular or square matrix A is said to be invertible and $A^{-1} = \frac{1}{|A|}$ Adj.A

***System of Linear Equations** :

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$.

$\begin{bmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$; $\{ |A| \neq 0\}$.

***Criteria of Consistency**.

(i) If $|A| \neq 0$, then the system of equations is said to be consistent & has a unique solution.

(ii) If $|A| = 0$ and (adj. A)B = 0, then the system of equations is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and (adj. A)B$\neq 0$, then the system of equations is inconsistent and has no solution.

Ch – 5 Continuity and Differentiability

Continuity and differentiability, chain rule, derivative of inverse trigonometric functions, like $sin^{-1}x$, $cos^{-1}x$, $tan^{-1}x$, derivative of implicit functions. Concept of exponential and logarithmic functions. Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.

Formulae for Limits

(a) $\lim_{x\to 0} cosx = 1$, (b) $\lim_{x\to 0} \frac{\sin x}{x} = 1$, (c) $\lim_{x\to 0} \frac{\tan x}{x} = 1$, (d) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(e) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$ (f) $\lim_{x\to 0} \frac{a^x - 1}{x} = log_{e^a}, a > 0$ (g) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (h) $\lim_{x\to 0} \frac{log_{e(1+x)}}{x} = 1$

(i)) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Important terms and facts about Limits & Continuity of a function

(i) For a function $f(x)$, $\lim_{x\to m} f(x)$ exists if $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x)$

(ii) A function $f(x)$ is continuous at a point $x = m$ if,

$$\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x) = f(m)$$

Where $\lim_{x\to m^-} f(x)$ *or* $\lim_{h\to 0} f(m-h)$ is **Left Hand Limit** of $f(x)$ at $x = m$ and

$\lim_{x\to m^+} f(x)$ *or* $\lim_{h\to 0} f(m+h)$ is **Right Hand Limit** of $f(x)$ at $x = m$ (0<h<<)

Also $f(m)$ is the value of the function $f(x)$ at $= m$.

Formulae for Derivatives

(i) $\frac{d}{dx}(x^n) = n\,x^{n-1}$, $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$, $\frac{d}{dx}\left(\sqrt{x}\right) = -\frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}(x) = 1$

(iii) $\frac{d}{dx}(c) = 0, \forall\, c \in R$

(iv) $\frac{d}{dx}(a^x) = a^x \log a,\ a > 0,\ a \neq 1$.

(v) $\frac{d}{dx}(e^x) = e^x$.

(vi) $\frac{d}{dx}(\log_a x) =, \frac{1}{x \log a}\ a > 0,\ a \neq 1,\ x$

(vii) $\frac{d}{dx}(\log x) = \frac{1}{x},\ x > 0$

(viii) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x \log a},\ a > 0,\ a \neq 1,\ x \neq 0$

(ix) $\frac{d}{dx}(\log |x|) = \frac{1}{x},\ x \neq 0$

(x) $\frac{d}{dx}(\sin x) = \cos x, \forall\, x \in R$.

(xi) $\frac{d}{dx}(\cos x) = -\sin x, \forall\, x \in R$.

(xii) $\frac{d}{dx}(\tan x) = \sec^2 x, \forall\, x \in R$.

(xiii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, \forall\, x \in R$.

(xiv) $\frac{d}{dx}(\sec x) = \sec x \tan x, \forall\, x \in R$.

(xv) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, \forall\, x \in R$.

(xvi) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

(xvii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

(xviii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \forall\ x \in R$

(xix) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \forall\ x \in R$.

(xx) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$.

(xxi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$.

(xxii) $\frac{d}{dx}(|x|) = \frac{x}{|x|},\ x \neq 0$

(xxiii) $\frac{d}{dx}(ku) = k\frac{du}{dx}$

(xxiv) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(xxv) $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

(xxvi) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

SOME ILUSTRATIONS :

****Q**. If $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11 & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$, continuous at x = 1, find the values of a and b.

Sol. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$.........(i)

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}[5a(1-h)-2b] = 5a-2b$

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[3a(1+h)+b] = 3a+b$

$f(1) = 11$

From (i) $3a+b = 5a-2b = 11$ and solution is $a = 3, b = 2$

Q. Find the relationship between a and b so that the function defined by $f(x) = \begin{cases} ax+1 & ,\text{if } x \le 3 \\ bx+3 & ,\text{if } x > 3 \end{cases}$ is continuous at x = 3.

Sol. $\because$ f(x) is cont. at x = 3 $\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$.........(i)

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}[a(3-h)+1] = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}[b(3+h)+3] = 3b+3$

$f(3) = 3a+1$

From (i) $3a+1 = 3b+3 = 3a+1 \Rightarrow 3a+1 = 3b+3$

$\Rightarrow$ $\mathbf{3a-3b=2}$ is the required relation between a and b

**If $y = (\log_e x)^x + x^{\log_e x}$ find $\frac{dy}{dx}$.

Sol. $y = (\log_e x)^x + x^{\log_e x} = e^{\log\{(\log_e x)^x\}} + e^{\log\{x^{\log_e x}\}}$

$= e^{x\log\{(\log_e x)\}} + e^{\log_e x.\log_e x}$

$\frac{dy}{dx}. = e^{x\log\{(\log_e x)\}}\left[x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1\right] + e^{\log_e x.\log_e x}\left[\frac{\log x}{x} + \frac{\log x}{x}\right]$

$= (\log_e x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log_e x}\left[2\frac{\log x}{x}\right]$

**If $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$

Sol. $x = a(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta)$

$y = a(1+\cos\theta) \Rightarrow \frac{dy}{d\theta} = a(-\sin\theta)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(-\sin\theta)}{a(1-\cos\theta)} = -\frac{2\sin\theta/2.\cos\theta s\theta}{2\sin^2\theta/2} = -\cot\frac{\theta}{2}$

$\frac{d^2y}{dx^2} = \text{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx} = \frac{1}{2}\text{cosec}^2\frac{\theta}{2}.\frac{1}{a(1-\cos\theta)}$

$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{2}\text{cosec}^2\frac{\pi}{4}.\frac{1}{a\left(1-\cos\frac{\pi}{2}\right)} = \frac{1}{2}.2.\frac{1}{a} = \frac{1}{a}$

Applications of derivatives: rate of change of quantities, increasing/decreasing functions, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

RATE OF CHANGE OF QUANTITIES

$\frac{dy}{dx}$ represent the rate of change of y w.r.t. x.

Also $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y w.r.t. x. at $x = x_0$.

Marginal cost is the derivative of cost function i.e. MC = $\frac{d}{dx}\{C(x)\}$.

Marginal revenue is the derivative of revenue function i.e., MR = $\frac{d}{dx}\{R(x)\}$.

INCREASING AND DECREASING FUNCTIONS

DEFINITION

Let I be an interval contained in the domain of a real valued function f .Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I => $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in I$.
(ii) Strictly increasing on I if $x_1 < x_2$ in I => $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
(iii) decreasing on I if $x_1 < x_2$ in I => $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.
(iv) Strictly decreasing on I if $x_1 < x_2$ in I => $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.

THEOREM: Let f be continuous on $[a, b]$ and differentiable on open interval (a, b) Then

(i) f is strictly increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in$ (a, b)
(ii) f is strictly decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in$ (a, b)
(iii) f is constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in$ (a, b)

MAXIMA AND MINIMA

Steps to solve a problem of maxima and minima

i) Formulate the given problem and find out $y = f(x)$.
ii) Find $\frac{dy}{dx}$.
iii) Equate $\frac{dy}{dx}$. to zero and find the solution of $\frac{dy}{dx} = 0$ .Let it is $c_1, c_2, c_3, \ldots\ldots$..these points are called *CRITICAL POINTS.* Now we have to check which point will be point of maxima and which will be point of minima. For this purpose we can use First derivative test or Second derivative test

For first derivative test

Draw a number line and put critical points

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, then c is a point of local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima infact such a point is called point of inflection.

For second derivative test

find out second order derivative i.e $\frac{d^2y}{dx^2}$.

If $\frac{d^2y}{dx^2} < 0$ at a critical point say c_1 then c_1 will be point of maxima and the corresponding value will be maximum value.

If $\frac{d^2y}{dx^2} > 0$ at a critical point say c_2 then c_2 will be point of minima and the corresponding value will be minimum value.

Ch – 7 Integration

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{px + q}{ax^2 + bx + c}dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx, \int \sqrt{a^2 \pm x^2}\, dx, \int \sqrt{x^2 - a^2}\, dx$$

$$\int \sqrt{ax^2 + bx + c}\, dx,$$

Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

INTEGRATION AS AN INVERSE PROCESS OF DIFFERENTIATION

1. If f(x) is derivative of function g(x), then g(x) is known as antiderivative or integral of f(x).

   i.e., $\frac{d}{dx}\{g(x)\} = f(x) \Leftrightarrow \int f(x)dx = g(x) + c$

2. Derivative of a function is unique but a function can have infinite antiderivatives or integrals.

3. $\int f(x)dx = g(x) + c$ is known as indefinite integral, where C is constant of integration.

4. $\int dx = x + c.$

5. $\int c.f(x)dx = c\int f(x)dx.$

6. $\int\{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx.$

7. $\int x^n\, dx = \frac{x^{n+1}}{n+1}, n \neq -1, n\ is\ a\ rational\ number. If\ n = 1, then \int \frac{1}{x}dx = \log|x| + c.$

8. $\int(ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$, n≠-1, is a rational number.

   If n=-1, then $\int \frac{1}{ax+b}dx = \frac{1}{a}log|ax + b| + c.$

9. $\int \sin ax\, dx = \frac{-\cos ax}{a} + c. If\ a = 1, \int \sin x\, dx = -\cos x + c.$

10. $\int \cos ax\, dx = \frac{\sin ax}{a} + c. If\ a = 1, \int \cos x\, dx = \sin x + c.$

11. $\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + c. If\ a = 1 . then\ \int \sec x \tan x\, dx = \sec x + c.$

12. $\int \text{cosec}\, ax \cot ax\, dx = \frac{1}{a}\text{cosec}\, ax + c. If\ a = 1 . then\ \int \text{cosec}\, x \cot x\, dx = -cosec\ x + c.$

13. $\int sec^2 ax\, dx = \frac{1}{a}\tan ax + c. If\ a = 1, then \int sec^2 x\, dx$= tan x+c.

14.$\int cosec^2 ax\, dx = \frac{1}{a}\cot ax + c. If\ a = 1, then \int cosec^2 x\, dx$= -cot x+c.

15.$\int e^{ax} dx = \frac{e^{ax}}{a}$ +c. If a=1 then $\int e^x dx = e^x$+c.

16.$\int a^{mx}\, dx = \frac{a^{mx}}{m log_e a}$+c. If m=1 then $\int a^x\, dx = \frac{a^x}{log_e a}$+c.

INTEGRATION BY SUBSTITUTION

1.$\int f(x)dx \Leftrightarrow \int f\{g(x)\}g'(x)dt$,if we substitute x =g(t) such that dx=g'(t)dt.

2.$\int \tan a\,x dx = -\frac{1}{a} log|\cos ax| + c\ or\ \frac{1}{a} log|\sec ax|$+c.

3.$\int \cot x\, dx$ =log$|\sin x|$+c.

4.$\int \sec x dx = log|\sec x + \tan x|$+c.

5.$\int cosec\ x\, dx = log|cosec\ x - \cot x|$+c.

INTEGRATION BY PARTIAL FRACTIONS

1. We must check that we are dealing with polynomials and degree of numerator is less than the degree of denominator and proceed for partial fraction. If not, divide numerator by denominator and write

$\frac{\text{Numerator}}{Denominator}$= quotient+$\frac{Remainder}{Denomiator}$ and proceed for partial fraction of $\frac{Remainder}{Denomiator}$.

i) when factor in denominator is linear and non-repeated.

$\frac{p(x)}{(x+a)(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}.$

ii) When factor in denominator linear and repeated.

$\frac{p(x)}{(x-a)^2(x+b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x+b}.$

iii) When factor in denominator is quadratic and non-repeated.

$$\frac{p(x)}{(x+a)(x^2+b)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b}.$$

2.To evaluate$\int \frac{ax+b}{px^2+qx+r} dx$. We write ax+b=$\lambda$(p$x^2$+qx+r) +$\mu$.

3. To evaluate $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}$dx.

Let x^2=y and proceed for partial fraction of $\frac{y}{(y+a^2)(y+b^2)}$ and after getting the partial fraction replace value of y as x^2 and then integrate.

INTEGRATION BY PARTS

Sometimes we get product of the functions which we cannot simplify in such cases we apply integration by parts. First check functions are in proper forms otherwise first reduce in proper form using substitution.

1. $\int\{f(x).g(x)\}dx = f(x).\int g(x)dx - \int\{\frac{d}{dx}f(x).\int g(x)dx\}dx.$

We can choose first and second functions according to ILATE, where I→Inverse trigonometrical function, L→Logarithmic function, A→Algebraic function, T→Trigonometric function and E→Exponential function.

DEFINITE INTEGRALS

$$\frac{d}{dx}\{g(x)\} = f(x) \Rightarrow \int_a^b f(x)dx = g(b) - g(a).$$

SOME PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^a f(x)dx = 0.$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b.$

4. $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx.$

5. $\int_0^a f(x)dx = \int_0^a f(a - x)dx.$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

7. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, if\ f(2a - x) = f(x) and\ \int_0^{2a} f(x)dx = 0, if\ f(2a - x) = -f(x).$

8. $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, if\ f(x) is\ an\ even\ function, i.e., f(-x) = f(x)$

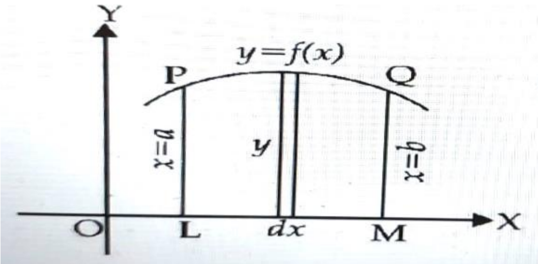
$$\int_{-a}^{a} f(x)dx = 0. if\ f(x) is\ odd\ function\ i.e., f(-x) = -f(x).$$

Ch – 8 Application of Integrals

Applications in finding the area under simple curves, especially lines, circles/ parabolas/ellipses (in standard form only)

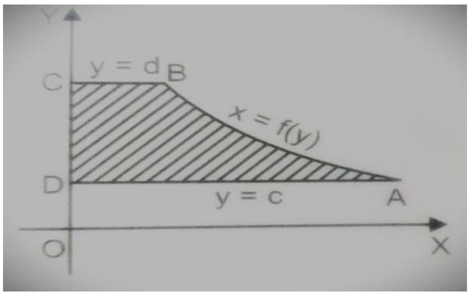
- One of the major applications of integrals is in determining the area under the curves.

1. Consider a function y = f(x), above the x-axis, between the ordinates x = a and x = b then the area is given as

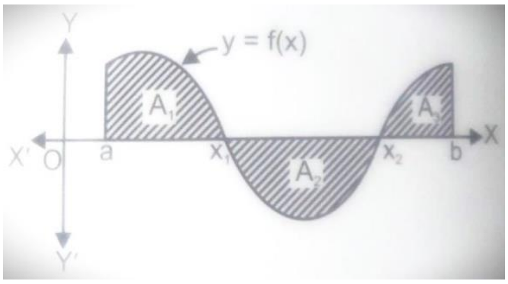


The area of the region $PQML = \int_a^b ydx = \int_a^b f(x)dx$

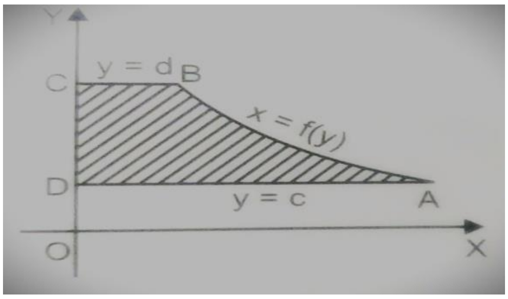
2. If the area is below the x-axis, then $A = \int_a^b f(x)dx$ is negative. Then the area is its magnitude. $\therefore A = \int_a^b -f(x)dx.$



3. If the curve $y = f(x)$ crosses x-axis to a number of times, then the area between the curve $y = f(x)$, and the ordinates x = a and x = b then the area is given as$A = A_1 + A_2 + A_3 + \cdots$



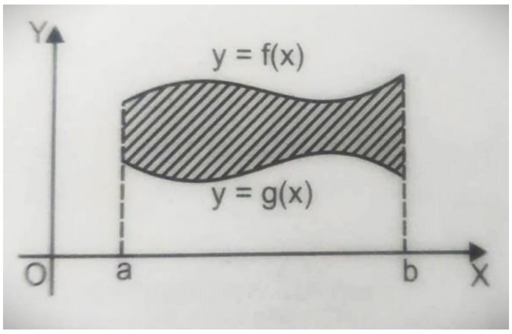
4. The area bounded by the curve x = f(y) and the abscissa y = c and y = d is to the right of y-axis is $A = \int_c^d xdy.$



5. The area bounded by the curve x = f(y) and the abscissa y = c and y = d is to the left of y-axis is $A = \int_c^d -x dy.$

6. Consider the two curves having equation of f(x) and g(x), the area between the region a, b of the two curves is given as

   dA = [f(x) – g(x)] dx, and the total area A can be taken as $Area = \int_a^b [f(x) - g(x)]dx$



Definition, order, and degree, general and solutions of a differential equation. Solution of differential equations by method of separation of variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$$\frac{dy}{dx} + py = q, \text{where p and q are functions of x or constants.}$$

$$\frac{dx}{dy} + px = q, \text{where p and q are functions of y or constants.}$$

Definitions:

An equation involving independent variable (variables), dependent variable and derivative or derivatives of dependent variable with respect to independent variable (variables) **is called a differential equation.**

Eg. $\frac{dy}{dx} - \sin x = 0$ and $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right) = 0$ are differential equations, but 2x-3y=0 is not a differential equation as derivative or dependent variable(y) with respect to independent variable(x) is not present.

Order of a Differential equation

The order of the highest order derivative of dependent rankle with respect to independent variable involved in a differential equation, is called order of differential equation.

e. g. (i) $\frac{dy}{dx} - \sin x = 0$ (ii) $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right) = 0$

Here, in e.g (i), equation has the highest derivative of first order and in e.g. (ii), equation has the highest derivative of third order. So, orders of the differential equations in e.g (i) and (ii) are I and 3, respectively.

Degree of a Differential Equation

The highest power (positive integral index) of the highest order derivative involved in a differential equation, when it is written as a polynomial in derivatives, is called the degree of a differential equation.

e. g. $\left(\frac{d^{3y}}{dx^3}\right)^2 + x\left(\frac{d^3y}{dx^3}\right) + 3y\left(\frac{dy}{dx}\right) = 0$

In this, the highest order derivative is $\frac{d^3y}{dx^3}$ whose highest power is 2. So, degree of differential equation is 2.

Solution of a Differential Equation

Suppose a differential equation is given to us, in which y is dependent variable and x is independent
variable. Then the function □ □x□ will be its solution. If it satisfies the given differential equation, i.e.

when the function □ is substituted for the unknown y (dependent variable) in the given differential equation. LHS becomes equal to RHS. The solution of a differential equation is of two types, which are given below.

General Solution of a Differential Equation

If the solution of the differential equation of order n contains n arbitrary constants, then it is called a general solution. eg. The general solution of $\frac{d^2y}{dx^2}+y=0$ is $A\cos x+BSinx$. But $y=A\cos x+\sin x$ and $y=\cos x+B\sin x$ is not the solution of given differential equation, as it contains only one arbitrary constant.

Particular Solution of a Differential Equation

The solution of a differential equation obtained by giving Particular values to the arbitrary constants in thegeneral solution, is called the particular solution. Other words, the solution free from arbitrary constant is called particular solution.

The general solution of $\frac{d^2y}{dx^2}+y=0$ is $y=A\cos x+\sin x$. If A=B=1, then $y=\cos x+\sin x$ is a particular solution of the given differential equation.

Differential Equation with Variables Separable

A order and first degree differential equation $\frac{dy}{dx}=F(x,y)$ is in the form of variable separable, if the function F can be expressed as the product of the functions of x and the functions of y Suppose a first order and first degree differential equation is given to us, i.e. $\frac{dy}{dx}=F(x,y)$.

Now , expressed it as $\frac{dy}{dx}=h(y),g(x),if h(y)\neq 0$ it can be written as $\frac{1}{h(y)}dy=g(x)dx$.

Integrate both sides

$$\int \frac{1}{h(x)}dy=\int g(x)dx$$

It is the required solution of given differential equation.

Homogeneous Differential Equation

A function F(x,y) is said to be homogeneous of degree n, if $F(x,y)=x^n g\left(\frac{y}{x}\right) or\ \ y^n h\left(\frac{x}{y}\right)$

A differential equation of the form $\frac{dy}{dx}=F(x,y)$ is called a homogeneous differential equation, if F(x,y) is a homogeneous function of degree zero.

Linear Differential Equation

A differential equation of the form $A_0\frac{d^ny}{dx^n}+A_1\frac{d^{n-1}}{dx^{n-1}}+A_2\frac{d^{n-2}y}{dx^{n-2}}+.......+A_{n-1}\frac{dy}{dx}+A_n y=0$ where $A_0,A_1,A_2....A_{n-1},A_n$ are either constants or functions of independent variable x, is called a linear differential equation.

Linear Differential Equation of First Order

A first order differential equation in which the degree of dependent variable and its derivative is one and they do not get multiplied together, is called a linear differential equation of first order.

A differential equation of the form $\frac{dy}{dx}+Py=Q$, Where P and Q are constants of x only. e.g $\frac{dy}{dx}+2y=\sin x$, P = 2 and $Q=\sin x$

Integrating Factor (IF):

Linear differential equations are solved when they are multiplied by a factor, which is called the integrating factor, because by multiplying such factor the left hand side of the differential equationbecome exact differential of same function.

For differential equation,

$$\frac{dy}{dx}+Py=Q$$

$$IF={}_{e}\int pdx$$

Now, use the formula, $y-IF=\int(Q.F)dx+C$.

Vectors and scalars, magnitude, and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in given ratio. Definition, Geometrical Interpretation, properties, and application of scalar (dot) product of vectors, vector (cross) product of vectors.

Important Points

Vector: A directed line segment is called a Vector. It has a magnitude and direction. Any number is called a scalar.

If $\vec{a} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$, then $|a| = \sqrt{a^2 + b^2 + c^2}$

Unit Vector*:* A vector with unit length along any vector $\vec{a}$ is called a unit vector in the direction of a and it is denoted by $\hat{a}$

Thus $\hat{a} = \frac{\vec{a}}{|a|}$

Collinear Vector*:* Two or more non-zero vectors are said to be collinear if they are parallel to the same line. Two vectors $\vec{a}$ and $\vec{b}$ are collinear if $\vec{a} = \gamma\,\vec{b}$ for some scalar γ.

Position Vector of a Point: Let O be the origin and OA= $\vec{a}$ we say that the position vector of A is $\vec{a}$

$\overrightarrow{AB}$=(Position vector of B) – (Position vector of A)

Section Formula: Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and let P be a point which divides AB in the ratio m: n. Then position vector of

P(r) = $\frac{m\vec{b}+n\vec{a}}{m+n}$

Component of a Vector: If $\vec{A}$= a $\hat{\imath}$ + b $\hat{\jmath}$ + c $\hat{k}$ we say that the component of $\vec{A}$ along X-axis, Y- axis and Z-axis are a,b,c respectively.

Dot or Scalar Product*:*

If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\vec{a}.\vec{b} = |a|\,|b|\cos\theta$

Vector Projection of $\vec{a}$ on $\vec{b} = \left(\frac{\vec{a}.\vec{b}}{|b|^2}\right)\vec{b}$

Cross or Vector Product:

If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\vec{a}$x$\vec{b} = |a|\,|b|\sin\theta\,\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

Area of a parallelogram with sides $\vec{a}$ and $\vec{b}$ = $|\vec{a}x\vec{b}|$

Area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ $= \frac{1}{2}\left|\vec{a} x \vec{b}\right|$

Area of a Δ ABC = $\frac{1}{2}\left|\overrightarrow{AB}\ X\ \overrightarrow{AC}\ \right|$

Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, skew lines, shortest distance between two lines. Angle between two lines.

Key Points

DIRECTION COSINES - The direction cosines of a line are defined as the direction cosines of any vector whose support is the given line.

A(x_1,x_2,x_3) B(y_1,y_2,y_3)

If A(x_1,x_2,x_3) and B(y_1,y_2,y_3) are two point on a line L, then its direction cosines are $< \frac{y_1-x_1}{AB}, \frac{y_2-x_2}{AB}, \frac{y_3-x_3}{AB} >$ where AB= distance between points A and B = $|\overrightarrow{AB}|$

In vector if $\overrightarrow{AB} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is position vector. The angle α, β and γ made by the vector $\vec{r}$=($\overrightarrow{AB}$) with the positive directions of x,y and z axes respectively are called its ***direction angles*** the cosine values of these angles $\cos\alpha,\ \cos\beta$ and $\cos\gamma$ are called direction cosines of $\overrightarrow{AB}$ usually denoted by l, m and n.

If $\overrightarrow{AB} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $|\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2}$

Direction cosines are given as follows:

$$l = \cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}},\quad m = \cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}},\quad n = \cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Relation between l, m & n is : $l^2 + m^2 + n^2=1$

DIRECTION RATIOS: - The direction ratios of a line are proportional to the direction ratios of any vector whose support is the given line.

If A (x_1,y_1,z_1) and B (x_2,y_2,z_2) are two points on a line, then its direction ratios are proportional to x_2-x_1, y_2-y_1, z_2-z_1.

A (x_1,y_1,z_1) B (x_2,y_2,z_2)

If $\overrightarrow{AB} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then d.r are x, y and z.

NOTE-Direction ratios (d.r) and direction cosines (d.c) of a line passing through two points A (x_1,y_1,z_1) and B (x_2,y_2,z_2) .

A (x_1,y_1,z_1) B (x_2,y_2,z_2)

d. r's are $< x_2-x_1,\ y_2-y_1,\ z_2-z_1 >$

D.C's are $< \frac{x_2-x_1}{AB}, \frac{y_2-y_1}{AB}, \frac{z_2-z_1}{AB} >$ Where AB=$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

EQUATION OF LINE IN DIFFERENT WAYS

1. Equation of a \`line passing through (x_1,y_1,z_1) and direction ratios < a,b,c>.
   is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

   A (x_1,y_1,z_1) d.r's < a,b,c>

2. Equation of a line passing through two given points A (x_1,y_1,z_1) and B (x_2,y_2,z_2)
   is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

   A (x_1,y_1,z_1) B (x_2,y_2,z_2)

3. Equation of a line through a given point (x_1,y_1,z_1) and parallel to a given vector $\overrightarrow{B} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

` A(x_1,x_2,x_3) D.r. a,b,c $\overrightarrow{B} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

4. If l,m,n are d.c. and a given point (x_1,y_1,z_1) of the line then the equation of the line is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

SKEW LINES: Two straight line in space which are neither parallel nor intersecting are called skew lines.

SHORTEST DISTANCE: The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q where the lines of shortest distance intersects the two given lines.

LINE OF SHORTEST DISTANCE: If l_1 and l_2 are two skew –lines ,then there is one and only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance. (1) If lines intersect then shortest distance between them is zero. (2) If lines are parallel then the shortest distance between them is the distance between the two lines.

SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Vector Form):-The shortest (S.D.) between two non-parallel lines $\vec{r} = \vec{a_1} + \alpha\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \alpha\,\vec{b_2}$ is given by

S.D. = $\left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Cartesian form)

If given equations in Cartesian form as below

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$ then for shortest distance

First we change this equation in vector form like as

$\vec{r} = \vec{a_1} + \alpha\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \beta\,\vec{b_2}$

Where $\vec{a_1} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$, $\vec{b_1} = l_1\hat{\imath} + m_1\hat{\jmath} + n_1\hat{k}$,

$\vec{a_2} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$, $\vec{b_2} = l_2\hat{\imath} + m_2\hat{\jmath} + n_2\hat{k}$

And use formula S.D. in vector form

S.D. = $\left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

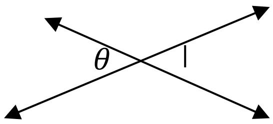
Shortest distance between the parallel lines $\vec{r} = \vec{a_1} + \alpha\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \beta\,\vec{b_1}$

S.D. = $\left|\frac{(\vec{a}_2-\vec{a}_1)\times\vec{b}_1}{|\vec{b}_1|}\right|$

ANGLE BETWEEN TWO LIES (VECTOR FORM)

$\vec{r} = \vec{a_1} + \alpha\,\vec{b_1}$ and $\vec{r} = \vec{a_2} + \beta\,\vec{b_2}$

COSθ = $\left|\frac{(\vec{b}_1.\vec{b}_2)}{|\vec{b}_1||\vec{b}_1|}\right|$



***ANGLE BETWEEN TWO LIES** (Cartesian form)*

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$

COSθ = $\left|\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}.\sqrt{l_2^2+m_2^2+n_2^2}}\right|$

Ch – 12 Linear Programming Problems

Introduction, related terminology such as constraints, objective function, optimization, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded or unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

TYPES OF QUESTIONS THAT MAY BE ASKED FROM THE UNIT

1. Questions related to the feasible region. (1m/2m)
2. Questions related to the feasible solutions. (1m/2m)
3. Question related to the optimal solution or optimal feasible solution. (1m/2m)
4. Question related to finding the optimal value. (1m/2m)
5. Questions related to bounded and unbounded region. (1m/2m)
6. Questions related to the formation of the constraints and objective function (2m/3m)
7. Questions related to solving the linear inequalities (constraints) graphically (3m/5m)
8. Questions related to solving LPP. (3m/5m)

PRE-REQUISITE KNOWLEDGE THAT WILL HELP IN BETTER REVISION OF THE UNIT

OBJECTIVE FUNCTION: The linear function that has to be optimized i. e. either to be maximized or minimized is called an objective function.

LINEAR CONSTRAINTS: The linear inequalities that represent the restrictions that are mentioned in the LPP are called the linear constraints.

NON-NEGATIVE RESTRICTIONS: $x \geq 0, y \geq 0$ are called the non negative constraints as the decision variables must be always greater than or equal to zero.

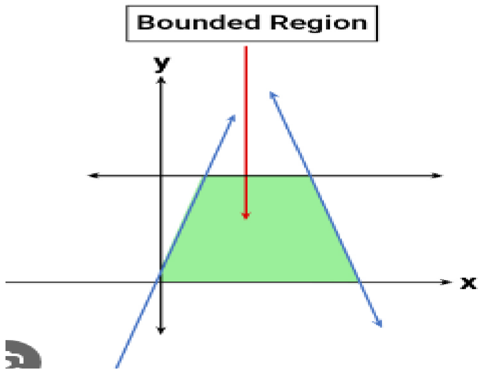
OPTIMAL SOLUTION: The point in the feasible region which optimizes the objective function is called optimal solution.

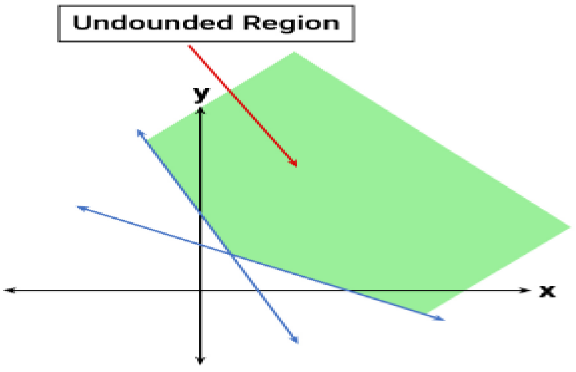
OPTIMAL VALUE: The maximum or the minimum value of the objective function which is obtained by putting the optimal solution in the objective function is called optimal value.

FEASIBLE REGION:

Bounded: A closed region which consists of all the possible solutions of the given system of constraint is called a feasible bounded region.

Unbounded: an open region which consists of all the possible solutions of the given system of constraints is called a feasible unbounded region.





FEASIBLE SOLUTIONS: A solution of a LPP which satisfy the non negativity restrictions of a problem is called its feasible solutions.

Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' theorem, Random variable, and its probability distribution, mean of random variable.

➢ *Conditional Probability*

If E and F are two events associated with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred, written as P (E | F), is given by:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

➢ *Properties of Conditional Probability*

Let E and F be events associated with the sample space S of an experiment. Then:

(i) P (S | F) = P (F | F) = 1

(ii) P [(A ∪ B) | F] = P (A | F) + P (B | F)-P [(A∩B | F)], where A and B are any two events associated with S.

(iii) P (E' | F) = 1-P (E | F)

➢ *Multiplication Theorem on Probability*

Let E and F be two events associated with a sample space of an experiment. Then

$$P(E \cap F) = \begin{cases} P(E)\,P(F \mid E), & P(E) \neq 0 \\ P(F)P(E \mid F), & P(F) \neq 0 \end{cases}$$

If E, F and G are three events associated with a sample space, then.

$$P(E \cap F \cap G) = P(E)\,P(F \mid E)\,P(G \mid E \cap F)$$

➢ *Independent Events*

Let E and F be two events associated with a sample space S. If the probability of occurrence of one of them is not affected by the occurrence of the other, then we say that the two events are independent. Thus, two events E and F will be independent, if

$P(F \mid E) = P(F)$, provided $P(E) \neq 0$ and $P(E \mid F) = P(E)$, provided $P(F) \neq 0$

Using the multiplication theorem on probability, we have $P(E \cap F) = P(E)\,P(F)$

Three events A, B and C are said to be mutually independent if all the following conditions hold.

$P(A \cap B) = P(A)\,P(B)$

$P(A \cap C) = P(A)\,P(C)$; $P(B \cap C) = P(B)\,P(C)$ and $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

➢ *Partition of a Sample Space*

A set of events $E_1, E_2, \ldots \ldots E_n$ is said to represent a partition of a sample space S if

(a) $E_i \cap E_j = \phi, i \neq j; i, j = 1,2,3 \ldots, n$

(b) $E_1 \cup E_2 \cup \ldots \cup E_n = S$ and

(c) Each $E_i \neq \phi, i.e., P(E_i) > 0 i = 1,2,3$ ..., n

- ***Theorem of Total Probability***

Let $\{E_1 E_2, \ldots, E_n\}$be a partition of the sample space S. Let A be any event associated with S, then

P(A) = $\sum_{j=1}^{n} P(E_j)P(A|E_j)$

- ***Bayes' Theorem***

If $E_1, E_2, \ldots \ldots E_n$ are mutually exclusive and exhaustive events associated with a sample space,

and A is any event of non-zero probability, then. $P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{i=1}^{n} P(E_i)P(A|E_i)}$

- ***Random Variable and its Probability Distribution***

A random variable is a real valued function whose domain is the sample space of a random experiment. The probability distribution of a random variable X is the system of numbers.

X:	x_1	x_2	$\cdots$	x_n
P(X):	p_1	p_2	$\cdots$	p_n

where, p_i > 0; i = 1, 2, ..., n, $\sum_{i=1}^{n} P_i = 1$.