DIRECTORATE OF EDUCATION
Govt. of NCT, Delhi

SUPPORT MATERIAL
(2022-2023)

Class : XII

MATHEMATICS

Under the Guidance of

Shri Ashok Kumar
Secretary (Education)

Shri Himanshu Gupta
Director (Education)

Dr. Rita Sharma
Addl. DE (School & Exam.)

Coordinators

Mr. Sanjay Subhas Kumar
DDE (Exam)

Mrs. Sunita Dua
OSD (Exam)

Mr. Raj Kumar
OSD (Exam)

Mr. Krishan Kumar
OSD (Exam)

Production Team
Anil Kumar Sharma

Published at Delhi Bureau of Text Books, 25/2, Institutional Area, Pankha Road, New Delhi-58 by **Rajesh Kumar**, Secretary, Delhi Bureau of Text Books and Printed at: Supreme Offset Press, 133, Udhyog Kendra Ext.-1, Greater Noida, U.P.

ASHOK KUMAR
IAS

सचिव (शिक्षा)
राष्ट्रीय राजधानी क्षेत्र
दिल्ली सरकार
पुराना सचिवालय, दिल्ली-110054
दूरभाष : 23890187 टेलीफैक्स : 23890119

सत्यमेव जयते

Secretary (Education)
Government of National Capital Territory of Delhi
Old Secretariat, Delhi-110054
Phone : 23890187 Telefax : 23890119
e-mail : secyedu@nic.in

MESSAGE

Remembering the words of John Dewey, "Education is not preparation for life, education is life itself, I highly commend the sincere efforts of the officials and subject experts from Directorate of Education involved in the development of Support Material for classes IX to XII for the session 2022-23.

The Support Material is a comprehensive, yet concise learning support tool to strengthen the subject competencies of the students. I am sure that this will help our students in performing to the best of their abilities.

I am sure that the Heads of School and teachers will motivate the students to utilise this material and the students will make optimum use of this Support Material to enrich themselves.

I would like to congratulate the team of the Examination Branch along with all the Subject Experts for their incessant and diligent efforts in making this material so useful for students.

I extend my Best Wishes to all the students for success in their future endeavours.

(Ashok Kumar)

HIMANSHU GUPTA, IAS
Director, Education & Sports

सत्यमेव जयते

Directorate of Education
Govt. of NCT of Delhi
Room No. 12, Civil Lines
Near Vidhan Sabha,
Delhi-110054
Ph.: 011-23890172
E-mail: diredu@nic.in

MESSAGE

"A good education is a foundation for a better future."
- Elizabeth Warren

Believing in this quote, Directorate of Education, GNCT of Delhi tries to fulfill its objective of providing quality education to all its students.

Keeping this aim in mind, every year support material is developed for the students of classes IX to XII. Our expert faculty members undertake the responsibility to review and update the Support Material incorporating thelatest changes made by CBSE. This helps the students become familiar with the new approaches and methods, enabling them to become good at problem solving and critical thinking. This year too, I am positive that it will help our students to excel in academics.

The support material is the outcome of persistent and sincere efforts of our dedicated team of subject experts from the Directorate of Education. This Support Material has been especially prepared for the students. I believe its thoughtful and intelligent use will definitely lead to learning enhancement.

Lastly, I would like to applaud the entire team for their valuable contribution in making this Support Material so beneficial and practical for our students.

Best wishes to all the students for a bright future.

(HIMANSHU GUPTA)

Dr. RITA SHARMA
Additional Director of Education
(School/Exam)

सत्यमेव जयते

Govt. of NCT of Delhi
Directorate of Education
Old Secretariat, Delhi-110054
Ph. : 23890185

D.O. No. PS/Addl.DE/Sch/2022/131
Dated: 01 सितम्बर, 2022

संदेश

शिक्षा निदेशालय, दिल्ली सरकार का महत्वपूर्ण लक्ष्य अपने विद्यार्थियों का सर्वांगीण विकास करना है। इस उद्देश्य को ध्यान में रखते हुए शिक्षा निदेशालय ने अपने विद्यार्थियों को उच्च कोटि के शैक्षणिक मानकों के अनुरूप विद्यार्थियों के स्तरानुकूल सहायक सामग्री कराने का प्रयास किया है। कोरोना काल के कठिनतम समय में भी शिक्षण अधिगम की प्रक्रिया को निर्बाध रूप से संचालित करने के लिए संबंधित समस्त अकादमि समूहों और क्रियान्वित करने वाले शिक्षकों को हार्दिक बधाई देती हूँ।

प्रत्येक वर्ष की भाँति इस वर्ष भी कक्षा 9वीं से कक्षा 12वीं तक की सहायक सामग्रियों में सी.बी.एस.ई के नवीनतम दिशा-निर्देशों के अनुसार पाठयक्रम में आवश्यक संशोधन किए गए हैं। साथ ही साथ मूल्यांकन से संबंधित आवश्यक निर्देश भी दिए गए हैं। इन सहायक सामग्रियों में कठिन से कठिन सामग्री को भी सरलतम रूप में प्रस्तुत किया गया है ताकि शिक्षा निदेशालय के विद्यार्थियों को इसका भरपूर लाभ मिल सके।

मुझे आशा है कि इन सहायक सामग्रियों के गहन और निंरतर अध्ययन के फलस्वरूप विद्यार्थियों में गुणात्मक शैक्षणिक संवर्धन का विस्तार उनके प्रदर्शनो में भी परिलक्षित होगा। इस उत्कृष्ट सहायक सामग्री को तैयार करने में शामिल सभी अधिकारियों तथा शिक्षकों को हार्दिक बधाई देती हूँ तथा सभी विद्यार्थियों को उनके उज्जवल भविष्य की शुभकामनाएं देती हूँ।

रीता शर्मा
(रीता शर्मा)

DIRECTORATE OF EDUCATION
Govt. of NCT, Delhi

SUPPORT MATERIAL

(2022-2023)

Class : XII

MATHEMATICS

(ENGLISH MEDIUM)

NOT FOR SALE

PUBLISHED BY : DELHI BUREAU OF TEXTBOOKS

भारत का संविधान

भाग 4क

नागरिकों के मूल कर्तव्य

अनुच्छेद 51 क

मूल कर्तव्य – भारत के प्रत्येक नागरिक का यह कर्तव्य होगा कि वह –

(क) संविधान का पालन करे और उसके आदर्शों, संस्थाओं, राष्ट्रध्वज और राष्ट्रगान का आदर करे;

(ख) स्वतंत्रता के लिए हमारे राष्ट्रीय आंदोलन को प्रेरित करने वाले उच्च आदर्शों को हृदय में संजोए रखे और उनका पालन करे;

(ग) भारत की संप्रभुता, एकता और अखंडता की रक्षा करे और उसे अक्षुण्ण बनाए रखे;

(घ) देश की रक्षा करे और आह्वान किए जाने पर राष्ट्र की सेवा करे;

(ङ) भारत के सभी लोगों में समरसता और समान भ्रातृत्व की भावना का निर्माण करे जो धर्म, भाषा और प्रदेश या वर्ग पर आधारित सभी भेदभावों से परे हो, ऐसी प्रथाओं का त्याग करे जो महिलाओं के सम्मान के विरुद्ध हों;

(च) हमारी सामासिक संस्कृति की गौरवशाली परंपरा का महत्त्व समझे और उसका परिरक्षण करे;

(छ) प्राकृतिक पर्यावरण की, जिसके अंतर्गत वन, झील, नदी और वन्य जीव हैं, रक्षा करे और उसका संवर्धन करे तथा प्राणिमात्र के प्रति दयाभाव रखे;

(ज) वैज्ञानिक दृष्टिकोण, मानववाद और ज्ञानार्जन तथा सुधार की भावना का विकास करे;

(झ) सार्वजनिक संपत्ति को सुरक्षित रखे और हिंसा से दूर रहे;

(ञ) व्यक्तिगत और सामूहिक गतिविधियों के सभी क्षेत्रों में उत्कर्ष की ओर बढ़ने का सतत् प्रयास करे, जिससे राष्ट्र निरंतर बढ़ते हुए प्रयत्न और उपलब्धि की नई ऊँचाइयों को छू सके; और

(ट) यदि माता-पिता या संरक्षक है, छह वर्ष से चौदह वर्ष तक की आयु वाले अपने, यथास्थिति, बालक या प्रतिपाल्य को शिक्षा के अवसर प्रदान करे।

Constitution of India

Part IV A (Article 51 A)

Fundamental Duties

It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

*(k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

Note: The Article 51A containing Fundamental Duties was inserted by the Constitution (42nd Amendment) Act, 1976 (with effect from 3 January 1977).

*(k) was inserted by the Constitution (86th Amendment) Act, 2002 (with effect from 1 April 2010).

भारत का संविधान

उद्देशिका

हम, भारत के लोग, भारत को एक [1][**संपूर्ण प्रभुत्व-संपन्न समाजवादी पंथनिरपेक्ष लोकतंत्रात्मक गणराज्य]** बनाने के लिए, तथा उसके समस्त नागरिकों को :

सामाजिक, आर्थिक और राजनैतिक **न्याय,**

विचार, अभिव्यक्ति, विश्वास, धर्म

और उपासना की **स्वतंत्रता,**

प्रतिष्ठा और अवसर की **समता**

प्राप्त कराने के लिए,

तथा उन **सब में**

व्यक्ति की गरिमा और [2][**राष्ट्र की एकता**

और अखंडता] सुनिश्चित करने वाली **बंधुता**

बढ़ाने के लिए

दृढ़संकल्प होकर **अपनी इस संविधान सभा में** आज तारीख 26 नवंबर, 1949 ई. को **एतद्द्वारा इस संविधान को अंगीकृत, अधिनियमित और आत्मार्पित करते हैं।**

---

1. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977 से) ''प्रभुत्व-संपन्न लोकतंत्रात्मक गणराज्य'' के स्थान पर प्रतिस्थापित।
2. संविधान (बयालीसवां संशोधन) अधिनियम, 1976 की धारा 2 द्वारा (3.1.1977 से) ''राष्ट्र की एकता'' के स्थान पर प्रतिस्थापित।

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

Team Members for Review of Support Material

S.No.	Name & Designation	School Name/Branch
1.	Dr. Gokula Nand Deolal (Principal)	OSD SOSE
2.	Mr. Vidya Sagar Malik (Lecturer Mathematics)	Core Academic Unit
3.	Mr. Udai Bir Singh (Lecturer Mathematics)	RPVV, Vasant Kunj
4.	Mr. Shashank Vohra (Lecturer Mathematics)	RPVV, Hari Nagar
5.	Smt. Suman Arora (Lecturer Mathematics)	RPVV, Paschim Vihar

ANNUAL SYLLABUS
MATHEMATICS (Code NO. 041)
Class-XII
Session 2022-23

One Paper **Max Marks:40**

No.	Units	No. of Periods	Marks
I.	Relations and Functions	30	08
II.	Algebra	50	10
III.	Calculus	80	35
IV.	Vectors and Three - Dimensional Gemetry	30	14
V.	Linear Programming	20	05
VI.	Probability	30	08
	Total	240	80
	Internal Assessment		20

Unit-I: Relations and Functions

1. Relations and Functions **15 Periods**

Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions.

2. Inverse Trigonometric Functions **15 Periods**

Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions.

Unit-II: Algebra

1. Matrices **25 Periods**

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operation on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Oncommutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants **25 Periods**

Determinant of a square matrix (up to 3 x 3 matrices), minors, co-factors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit-III: Calculus

1. Continuity and Differentiability **20 Periods**

Continuity and differentiability, chain rule, derivative of inverse trigonometric functions, like $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$, derivative of implicit functions. Concept of exponential and logarithmic functions.

Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.

2. Applications of Derivatives **10 Periods**

Applications of derivatives: rate of change of bodies, increasing/decreasing functions, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real life situations).

3. Integrals **20 Periods**

Integration as inverse process of differentiation. Integration of a variety of function by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{px + q}{ax^2 + bx + c}dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx, \int \sqrt{a^2 \pm x^2}dx, \int \sqrt{ax^2 + bx + c}\,dx$$

Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. Applications of the Integrals **15 Periods**

Applications in finding the area under simple curves, especially lines, circles/ellipses (in standard form only)

5. Differential Equations **15 Periods**

Definition, order and degree, general and particular solutions of a differential equation. Solution of differential equations by method of separation of variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$$\frac{dy}{dx} + py = q, \text{ where p and q are functions of x or constant.}$$

$$\frac{dx}{dy} + px = q, \text{ where p and q are functions of y or constant.}$$

Unit-IV: Vectors and Three-Dimensional Geometry

1. Vectors **15 Periods**

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors.

2. Three - dimensional Geometry **15 Periods**

Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, skew lines, shortest distance between two lines. Angle between two lines.

Unit-V: Linear Programming

1. Linear Programming **20 Periods**

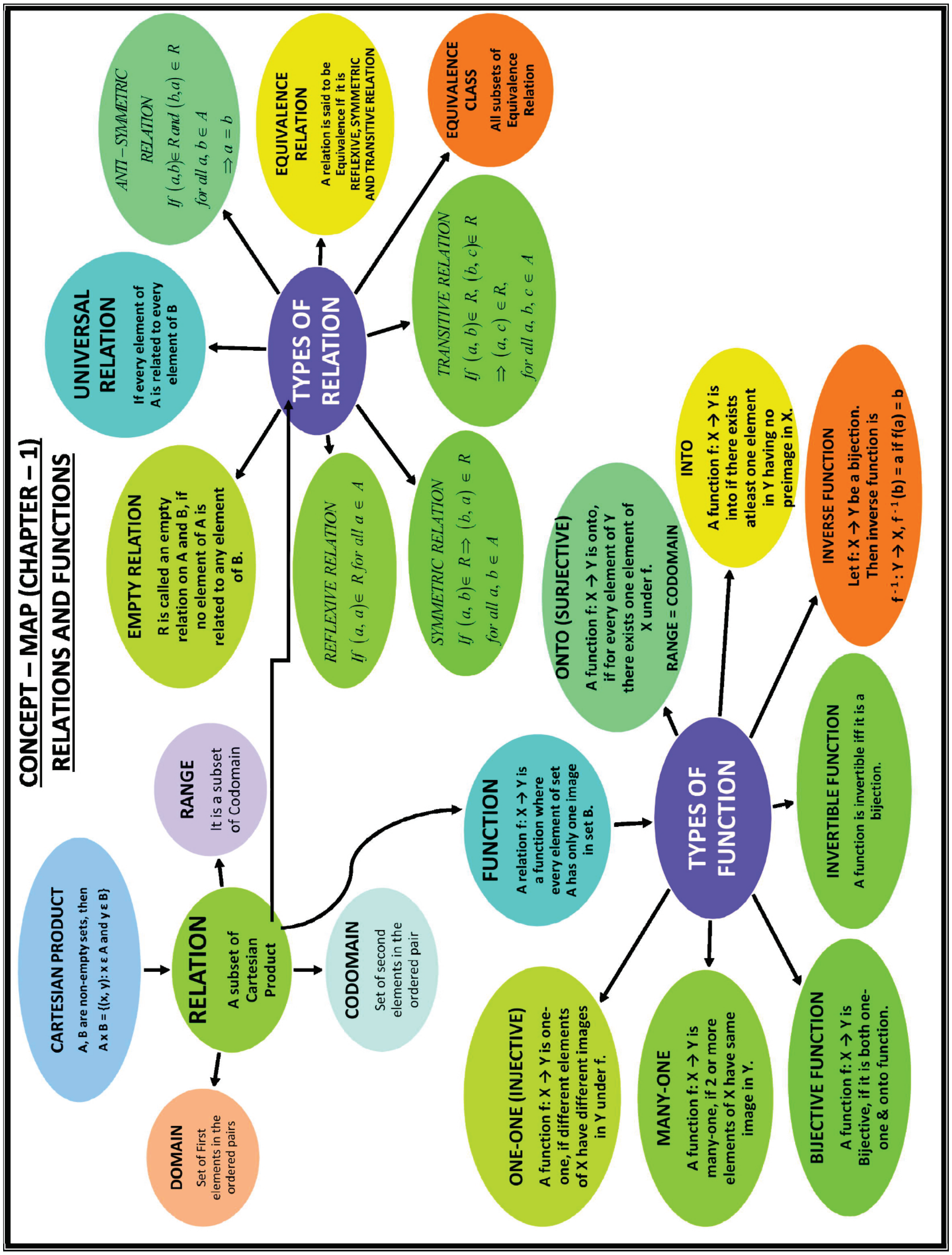
Introduction, related terminology such as constraints, objective function, optimization, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded or unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit-VI: Probability

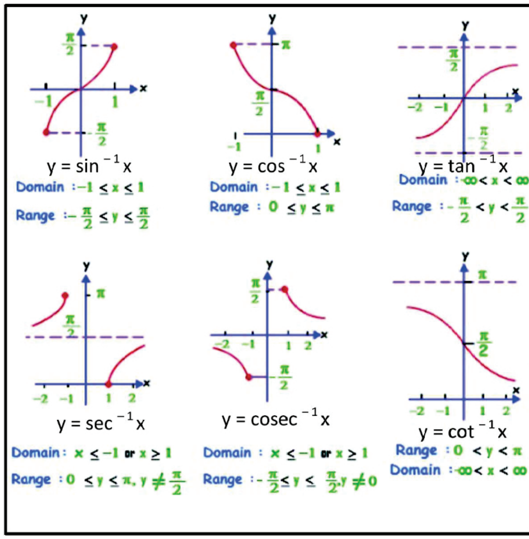
1. Probability

Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' theorem, Random variable and its probability distribution.

Contents


CONCEPT – MAP (CHAPTER – 1)
RELATIONS AND FUNCTIONS
CARTESIAN PRODUCT
A, B are non-empty sets, then A x B = {(x, y): x ε A and y ε B}
RELATION
A subset of Cartesian Product
DOMAIN
Set of First elements in the ordered pairs
RANGE
It is a subset of Codomain
CODOMAIN
Set of second elements in the ordered pair
TYPES OF RELATION
EMPTY RELATION
R is called an empty relation on A and B, if no element of A is related to any element of B.
UNIVERSAL RELATION
If every element of A is related to every element of B
ANTI – SYMMETRIC RELATION
If (a,b)∈ R and (b,a) ∈ R for all a, b ∈ A ⇒ a = b
EQUIVALENCE RELATION
A relation is said to be Equivalence if it is REFLEXIVE, SYMMETRIC AND TRANSITIVE RELATION
REFLEXIVE RELATION
If (a, a)∈ R for all a ∈ A
SYMMETRIC RELATION
If (a, b)∈ R⇒ (b, a) ∈ R for all a, b ∈ A
TRANSITIVE RELATION
If (a, b)∈ R, (b, c)∈ R ⇒ (a, c) ∈ R, for all a, b, c ∈ A
EQUIVALENCE CLASS
All subsets of Equivalence Relation
FUNCTION
A relation f: X → Y is a function where every element of set A has only one image in set B.
TYPES OF FUNCTION
ONE-ONE (INJECTIVE)
A function f: X → Y is one-one, if different elements of X have different images in Y under f.
MANY-ONE
A function f: X → Y is many-one, if 2 or more elements of X have same image in Y.
BIJECTIVE FUNCTION
A function f: X → Y is Bijective, if it is both one-one & onto function.
INVERTIBLE FUNCTION
A function is invertible iff it is a bijection.
ONTO (SURJECTIVE)
A function f: X → Y is onto, if for every element of Y there exists one element of X under f.
RANGE = CODOMAIN
INTO
A function f: X → Y is into if there exists atleast one element in Y having no preimage in X.
INVERSE FUNCTION
Let f: X → Y be a bijection. Then inverse function is
f⁻¹: Y → X, f⁻¹(b) = a if f(a) = b

CONCEPT – MAP (CHAPTER – 2)
INVERSE TRIGONOMETRIC FUNCTIONS



INVERSE TRIGONOMETRIC FUNCTIONS

- $\sin^{-1} x = \operatorname{cosec}^{-1}(\frac{1}{x}),\ when\ x \in [-1,1]$
- $\cos^{-1} x = \sec^{-1}(\frac{1}{x}),\ when\ x \in [-1,1]$
- $\tan^{-1} x = \cot^{-1}(\frac{1}{x}),\ when\ x \in R$
- $\cot^{-1} x = \tan^{-1}(\frac{1}{x}),\ when\ x \in R$
- $\sec^{-1} x = \cos^{-1}(\frac{1}{x}),\ when\ x \in R-(-1,1)$
- $\operatorname{cosec}^{-1} x = \sin^{-1}(\frac{1}{x}),\ when\ x \in R-(-1,1)$

- $\sin^{-1}(\sin x) = x,\ when\ x \in [-\frac{\pi}{2},\frac{\pi}{2}]$
- $\cos^{-1}(\cos x) = x,\ when\ x \in [0,\pi]$
- $\tan^{-1}(\tan x) = x,\ when\ x \in (-\frac{\pi}{2},\frac{\pi}{2})$
- $\cot^{-1}(\cot x) = x,\ when\ x \in (0,\pi)$
- $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x,\ when\ x \in [-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$
- $\sec^{-1}(\sec x) = x,\ when\ x \in [0,\pi]-\{\frac{\pi}{2}\}$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ when\ x \in [-1,1]$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ when\ x \in R$
- $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2},\ when\ x \in R-(-1,1)$

- $\sin(\sin^{-1} x) = x,\ when\ x \in [-1,1]$
- $\cos(\cos^{-1} x) = x,\ when\ x \in [-1,1]$
- $\tan(\tan^{-1} x) = x,\ when\ x \in R$
- $\cot(\cot^{-1} x) = x,\ when\ x \in R$
- $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x,\ when\ x \in R-(-1,1)$
- $\sec(\sec^{-1} x) = x,\ when\ x \in R-(-1,1)$

Inverse trigonometric functions are the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant and cosecant functions.

- $\sin^{-1}(-x) = -\sin^{-1}(x),\ when\ x \in [-1,1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1}(x),\ when\ x \in [-1,1]$
- $\tan^{-1}(-x) = -\tan^{-1}(x),\ when\ x \in R$
- $\cot^{-1}(-x) = \pi - \cot^{-1}(x),\ when\ x \in R$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x),\ when\ x \in R-(-1,1)$
- $\sec^{-1}(-x) = \pi - \sec^{-1}(x),\ when\ x \in R-(-1,1)$

CONCEPT – MAP (CHAPTER – 3)

MATRICES

MATRICES

Matrices are defined as a rectangular arrangement of numbers or functions,so it is 2-dimensional.

A two-dimensional matrix consists of the number of rows (m) and a number of columns (n). Horizontal ones are called Rows and Vertical ones are called Columns.

ORDER OF MATRIX

The order of matrix is a relationship with the number of elements present in a matrix.

The order of a matrix is denoted by $m \times n$, where m and n are the number of Rows and Columns Respectively and the number of elements in a matrix will be equal to the product of m and n.

Properties of transpose:

$(a)\ (A^T)^T = A$

$(b)\ (kA)^T = kA^T\ (k = \text{constant})$

$(c)\ (A + B)^T = A^T + B^T$

$(d)\ (AB)^T = B^T.A^T$

If $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}_{3\times 2}$ *then transpose of A donated* A^T *or* A'

and $A^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}_{2\times 3}$

$[\text{order of } A = m \times n, \text{ then order of } A^T = n \times m]$

Symmetric Matrix

A square matrix $A = [a_{ij}]$ *is symmetric if* $\boxed{A^T = A}$ i.e. $a_{ij} = a_{ji} \forall i$ *and* j

For Example:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

- *ROW MATRIX*, $A = (1\ \ 2\ \ 3)_{1\times 3}$
- *COLUMN MATRIX*, $A = \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{3\times 1}$
- *SQUARE MATRIX*, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$
- *ZERO / NULL MATRIX*

  $A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- *DIAGONAL MATRIX*, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

  (*Diagonal* elements may be zero)

- *SCALAR MATRIX*, $S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
- *IDENTITY MATRIX*, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Skew – Symmetric Matrix

A square matrix $A = [a_{ij}]$ is Skew-*Symmetric if* $\boxed{A^T = -A}$ i.e. $a_{ij} = -a_{ji} \forall i$ *and* j

(*All diagonal elements are zero*)

For Example:

$$A = \begin{pmatrix} 0 & -b & -c \\ b & 0 & e \\ c & -e & 0 \end{pmatrix}$$

CONCEPT – MAP (CHAPTER – 4)
DETERMINANTS

DETERMINANTS

Every square matrix associates to an expression or a number which is known as Determinant.

If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by $\det(A)$ or $|A|$ or Δ.

A determinant of order 2 is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

where a, b, c, d are complex numbers. The value of a determinant may be positive, negative or zero.

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$A = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| \text{ (sq. units)}$$

MINOR : If we take an element of the determinant and delete / remove the row and column containing that element, the determinant of the elements left is called the minor of that element.

It is denoted by Mij.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix}$$

(Minor of $a_{11} = M_{11}$)

PROPERTY : If we multiply the elements of any row / column with their respective Cofactors of the other row / Column, then we get zero as a result.

For example,

$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$

$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

COFACTOR : Cofactor of the element aij is $C_{ij} = (-1)^{i+j} M_{ij}$; Where i & j denotes the row & column in which the particular element lies

> Magnitude of Minor and Cofactor of a_{ij} are equal

$C_{12} = -M_{12}, C_{21} = -M_{21}, C_{23} = -M_{23}$

$C_{11} = M_{11}, C_{22} = M_{22}, C_{13} = M_{13}$

ADJOINT OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$ be a square matrix and C_{ij} be cofactor of a_{ij} in $|A|$. Then,

$(adj\ A) = [C_{ij}]$

- $A.(adj\ A) = (adj\ A).A = |A| I$
- $(adj\ AB) = (adj\ B).(adj\ A)$
- $|adj\ A| = |A|^{n-1}$,

where n is the order of a Matrix A

PROPERTY : If we multiply the elements of any row / column with their respective Cofactors of the same row / Column, then we get the value of the determinant.

For example,

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$|A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as : $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

OR

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

CONCEPT – MAP (CHAPTER – 4)

DETERMINANTS

DETERMINANTS

A square matrix A is said to be invertible if there exists a square matrix B of the same order such that $AB = BA = I$ then we write $A^{-1} = B$.

(A^{-1} exists only if $|A| \neq 0$)

$$A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

INVERSE OF 2×2 MATRIX

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus, If $A = \begin{pmatrix} 2022 & 1 \\ 2021 & 1 \end{pmatrix}$

then, $A^{-1} = \begin{pmatrix} 1 & -1 \\ -2021 & 2022 \end{pmatrix}$

SINGULAR MATRIX

A Matrix A is singular if $|A| = 0$ and it is non-singular if $|A| \neq 0$,

So, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5 \neq 0$

A is Non-Singular Matrix.

$|B| = \begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix} = 8 - 8 = 0,$

So B is Singular Matrix.

PROPERTIES OF A^{-1}:

- $(AB)^{-1} = B^{-1}.A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $A.A^{-1} = A^{-1}A = I$
- $|A^{-1}| = \frac{1}{|A|}$
- $|A.adjA| = |A|^n$

(Where n is the order of Matrix A)

Solution of system of Linear Equations Using Matrix method

Consider $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Then, we can write these equations as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \Rightarrow AX = B$$

Thus, Unique solution is given by $X = A^{-1}B$, when $|A| \neq 0$.

A system of equation $AX = B$ is said to be consistent or inconsistent according as its solution exists or not.

For a square matrix A in matrix equation $AX = B$

- If $|A| \neq 0$, there exists a unique solution and system of equations is consistent.
- If $|A| = 0$ & $(adjA).B \neq 0$, there exists no solution and system of equations is inconsistent.
- If $|A| = 0$ & $(adjA).B = 0$, then system may or may not be consistent according as the system has infinitely many solution or no solution.

5 CONCEPT MAP OF CONTINUITY AND DIFFERENTIABILITY

CONTINUITY

Continuity of a function at a point -
Suppose f is a rael function on a subset of real numbers & let c be a point in the domain of f . then f is continuous at c if $\lim_{x\to c} f(x)=f(c)$

Continuity of a function in an interval
Suppose f is a function defined on a closed interval [a,b], then for f to be continuous it needs to be continuous at every point in [a,b]includint the end points a & b
Continuity of f at a, $\lim_{x\to a^+} f(x)=f(a)$

Continuity of f at b, $\lim_{x\to b^-} f(x)=f(b)$

A function which is not continuous at x=c is said to be discontinuous at that point

Differentiation of Inverse Trigonometric Functions

$f(x)$	$f'(x)$	Domain of f
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$	(-1,1)
$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$	(-1,1)
$\tan^{-1}x$	$\frac{1}{1+x^2}$	R
$\cot^{-1}x$	$\frac{-1}{1+x^2}$	R
$sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert>1$
$cosec^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert>1$

Algebra of Continuous Function

Theorem 1:Suppose f & g be two functions continuous at a real numbe c , Then
(1)f+g is continuous at x=c
(2)f-g is continuous at x=c
(3)f.g is continuous at x=c
(4)f/g is continuous at x=c (provided g(c)≠0)
Theorem 2; Suppose f & g are real valued functions such that
Fog is defined at c .If g is continuous at c & if f is Continuous at g(c) the (fog) is continuous at c

Implicit Functions

An equation in the form f(x,y)=0 in which y is not expressible in terms of x is called implicit function of x &y

Derivative of Implicit Functions

Let y=f(x,y) where f(x,y) is an implicit function of x&y
*Firstly differentiate both sides of equation w.r.t x.
*Then take all terms involving dy/dx on LHS & remaining terms on R H S to get the required value .

Algebra Of Derivatives

Let u v be the function of x

(1)Sum and Differnce rule

$$(u\pm v)'=(u'\pm v')$$

(2) Leibnitz or product rule (uv)'=u'v+vu'

(3) Quotient Rule $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$

Differentiability

A function is said to be differentiable at a point c in its domain if its left hand And right hand derivatives exists at C and are equal
Here at x=c , left Hand Derivative L.H. D= $\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}=Lf'(c)$

Right Hand Derivative R.H.D= $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=Rf'(c)$

Theorem: If a fuction f is differntiable at a point c then it is also continuous at that point therefore every differentiable function is continuous but converse is not true

Chain Rule

If y is a function of u
u is a function of v, &
v is a function of x then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$$

Derivative of a Function in Parametric Form

The set of equations x=f(t), y=g(t) is Called parametric form of an equation
Here dy/dx=(dy/dt)/ dx/dt=g(t)/f(t)
Here dy/dx is expressed in terms of parameters only Without directly involving the main variable.

Second Order Derivative

Let y=f(x) then
If f(x) is differentiable
then we may differentiate it again w.r.t x & get the second order derivative represented by $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$
or f''(x) or D''yor y''or y_2

CONCEPT MAP OF CONTINUITY AND DIFFERENTIABILITY

Noteworthy Results on Continuous Functions
* A constant Function f(x)=k is continuous everywhere.
* Identity Function f(x)=x is continuous everywhere.
* Polynomial Function f(x)= $f(x)=a_0+a_1x+a_2x^2+..........a_nx^n, n\in N, x\in R$ is continuous everywhere.
* The modulus function f(x)=|x| is continuous everywhere.
* The logarithmic function f(x)=x is continuous in its domain
* The exponential function f(x)= $a^x, a>0$ is continuous everywhere.
* The sine function f(x)=sinx and cosine function f(x) =cosx are everywhere continuous .
*The tangent function, cotangent function, secant function and cosecant function are continuous in their respective domains.
*All the six inverse trigonometric functions are continuous in their respective domains.
*A rational function f(x)=g(x)/h(x) , h(x)not equal to zero is continuous at every point of its domain.
* Sum , difference ,product and quotient of of two continuous function is a continuous function.

A function f may fail to be continuous at x=0 for any of the following reasons

(1) f is not defined at x=a, i.e,f(a) does not exist

(2)Either $\lim_{x\to a^-} f(x)$ does not exist or $\lim_{x\to a^+} f(x)$ does not exist.

(3) $\lim_{x\to a^-} f(x)\neq\lim_{x\to a^+} f(x)$

(4) $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)\neq f(a)$

6 APPLICATION OF DERIVATIVE

Rate of Change of Quantities

If a quantity if 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx}\left[f'(x)\right]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_0} (f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_0$

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

For eg: if the radius of a circle, $r = 5$ cm, then the rate of change of the area of a circle per second w.r.t 'r' is – $\left.\frac{da}{dr}\right|_{r=5} = \frac{d}{dr}(\pi r^2)\Big|_{r=5} = 2\pi r\Big|_{r=5} = 10\pi$

Maxima & Minima

A point C in the domain of 'f' at which either $f'(c)=0$ or is not differentiable is called a critical point of f.

First Derivative Test

Let f be continuous at a critical point C in open I. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second Derivative Test

Let f be a function defined on I and C∈I, f is twice differentiable at C. Then
(i) $x=C$ is a point of local max. If $f'(C)=0$ and $f''(C) < 0$, $f(C)$ is local max. of f.
(ii) $x=C$ is a point of local min if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local min of f. **(iii)** The test fails if $f'(C)=0$ and $f''(C)=0$

Increasing & Decreasing Functions

A function f is said to be (i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \le f(x_2) \forall x_1, x_2 \in (a,b)$, and (ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in (a,b)$

If $f'(x) \ge 0 \forall x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \le 0 \forall x \in (a,b)$, then f is decreasing in (a,b) For eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in R$.
So, the function f is strictly increasing on R.

7 CONCEPT MAP OF INTEGRAL

INTEGRATION BY SUBSTITUTION

The method in which we change the variable to some other variable is called the method of substitution

$\int \tan x\,dx = \log|\sec x| + c$ $\int \cot x\,dx = \log|\sin x| + c$

$\int \sec x\,dx = \log|\sec x + \tan x| + c$ $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c.$

INDEFINITE INTEGRAL

It is the inverse of differentiation. Let $\frac{d}{dx}F(x) = f(x)$. Then $\int f(x)dx = F(x) + c$, 'c': constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$. (ii) $\int kf(x)dx = k\int f(x)dx$,

For eg: $\int (3x^2 + 2x)dx = x^3 + x^2 + c$ where k is real.

INTEGRATION OF SOME SPECIAL FUNCTIONS

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ (ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

(v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ (vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c.$

SOME STANDARD INTEGRALS

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$ like, $\int dx = x + c$

(ii) $\int \cos x\,dx = \sin x + c$ (iii) $\int \sin x\,dx = -\cos x + c$

(iv) $\int \sec^2 x\,dx = \tan x + c$ (v) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(vi) $\int \sec x \tan x\,dx = \sec x + c$ (vii) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ (ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ (xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$

(xii) $\int e^x dx = e^x + c$ (xiii) $\int a^x dx = \frac{a^x}{\log a} + c$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$ (xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$

(xvi) $\int \frac{1}{x}dx = \log|x| + c$

INTEGRATION BY PARTS

$$\int f_1(x) f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int \frac{d}{dx}f_1(x)\int f_2(x)\,dx$$

INTEGRATION BY PARTIAL FRACTIONS

A rational function of the form $\frac{P(x)}{Q(x)}(Q(x) \neq 0) = T(x) + \frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b.$

(ii) $\frac{px+q}{(x+a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

SOME SPECIAL TYPE OF INTEGRALS

(i) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c.$

(ii) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c.$

(iii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$

FIRST FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let the area function be defined by $A(x) = \int_a^x f(x)dx \; \forall x \geq a$, where f is continuous on $[a,b]$ then $A'(x) = f(x) \; \forall x \in [a,b]$.

SECOND FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f be a continuous function of x defined on $[a,b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x) \; \forall x \in$ domain of f, then $\int_a^b f(x)dx = [F(x) + c]_a^b = F(b) - F(a)$. This is called the definite integral of f over the range $[a,b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

INTEGRATION OF SOME SPECIAL FUNCTIONS

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2} \log \left| \frac{x-a}{x+a} \right|$ (ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

(iii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$ (iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + c$

(v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$ (vi) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

SOME STANDARD INTEGRALS

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ like, $\int dx = x + c$

(ii) $\int \cos x \, dx = \sin x + c$ (iii) $\int \sin x \, dx = - \cos x + c$

(iv) $\int \sec^2 x dx = \tan x + c$ (v) $\int \operatorname{cos} ec^2 x \, dx = - \cot x + c$

(vi) $\int \sec x \tan x \, dx = \sec x + c$ (vii) $\int \operatorname{coses} x \cot x \, dx = - \operatorname{cosec} x + c$

(viii) $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + c$ (ix) $\int \frac{dx}{\sqrt{1 - x^2}} = \cos^{-1} x + c$

(x) $\int \frac{dx}{1 + x^2} = \tan^{-1} x + c$ (xi) $\int \frac{dx}{1 + x^2} = \cos^{-1} x + c$

(xii) $\int e^x dx = e^x + c$ (xiii) $\int a^x dx = \frac{a^x}{\log a} + c$

(xiv) $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + c$ (xv) $\int \frac{dx}{x\sqrt{x^2 - 1}} = - \operatorname{cosec}^{-1} x + c$

(xvi) $\int \frac{1}{x} dx = \log |x| + c$

INTEGRATION BY PARTS

$$\int f_1(x) f_2(x) dx = f_1(x) \int f_2(x) dx - \int \frac{d}{dx} f_1(x) \int f_2(x) dx$$

INTEGRATION BY PARTIAL FRACTIONS

A rational function of the form $\frac{P(x)}{Q(x)}$ $(Q(x) \neq 0) = T(x) + \frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of Q(x). We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms:

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a \neq b$

(ii) $\frac{px+q}{(x+a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px+qx+r}{(x+a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x+a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(iv) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

INTEGRATION BY PARTIAL FRACTIONS

(i) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

(ii) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

(iii) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$.

FIRST FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let the area functions be defined by $A(x) = \int_a^x f(x)\,dx$ A $\geq a$, where f is continuous on [a, b] then A'(x) = f(x)x $\forall \in$ [a, b].

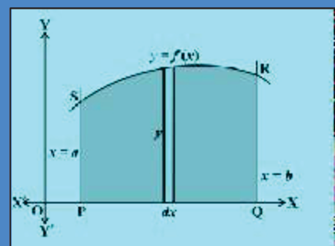
SECOND FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f be a continuous functions of x defined on [a, b] and let F be another function such that $\frac{d}{dx}$F(x) = f(x) $\forall\ x \in$ domain of f_1 then $\int_a^b f(x)\,dx = [F(x)+c]_a^b$ = F(b) – F(a). This a called the definite integral of f over the range [a, b] where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

CONCEPT – MAP (CHAPTER – 8)
APPLICATIONS OF INTEGRALS

APPLICATIONS OF INTEGRALS

Area of the regions bounded by simple curves

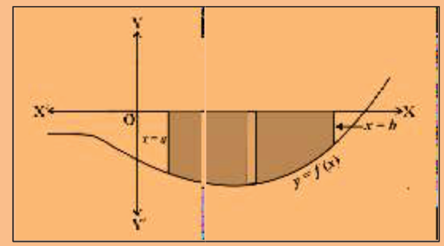


(A) *The area bounded by the curve* $y = f(x)$ *lies above the* X – *axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$Area = \int_a^b y dx = \int_a^b f(x) dx$$

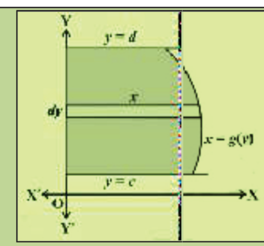
(B) *The area bounded by the curve* $y = f(x)$ *lies below the* X – *axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$Area = -\int_a^b y dx = \left| \int_a^b y \, dx \right| = \left| \int_a^b f(x) dx \right|$$



(C) *The area bounded by the curve* $x = f(y)$ *, lies right* Y – *axis and abscissae* $y = c$ *and* $y = d$ *is given by*

$$Area = \int_c^d x \, dy = \int_c^d g(y) dy$$

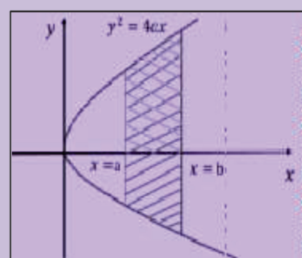


CONCEPT – MAP (CHAPTER – 8)
APPLICATIONS OF INTEGRALS

APPLICATIONS OF INTEGRALS

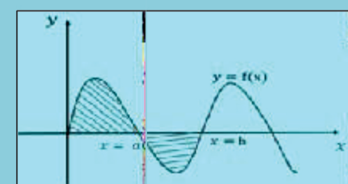
Symmetrical Area

If the curve is symmetrical about a coordinate axis (x axis , y axis , origin, a line), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.



Positive and Negative area

Positive and Negative area: Area is always taken as positive. If some part of the area lies in the + ve side i.e., above $X-$axis and some part lies in the $-$ve i.e., below $X-$axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.



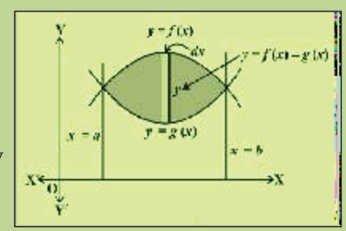
$$Area = \int_0^a y dx + \left| \int_a^b y\, dx \right|$$

Area between two curves

Area between two curves:

When both curves intersect at two points and their common area lies between these points

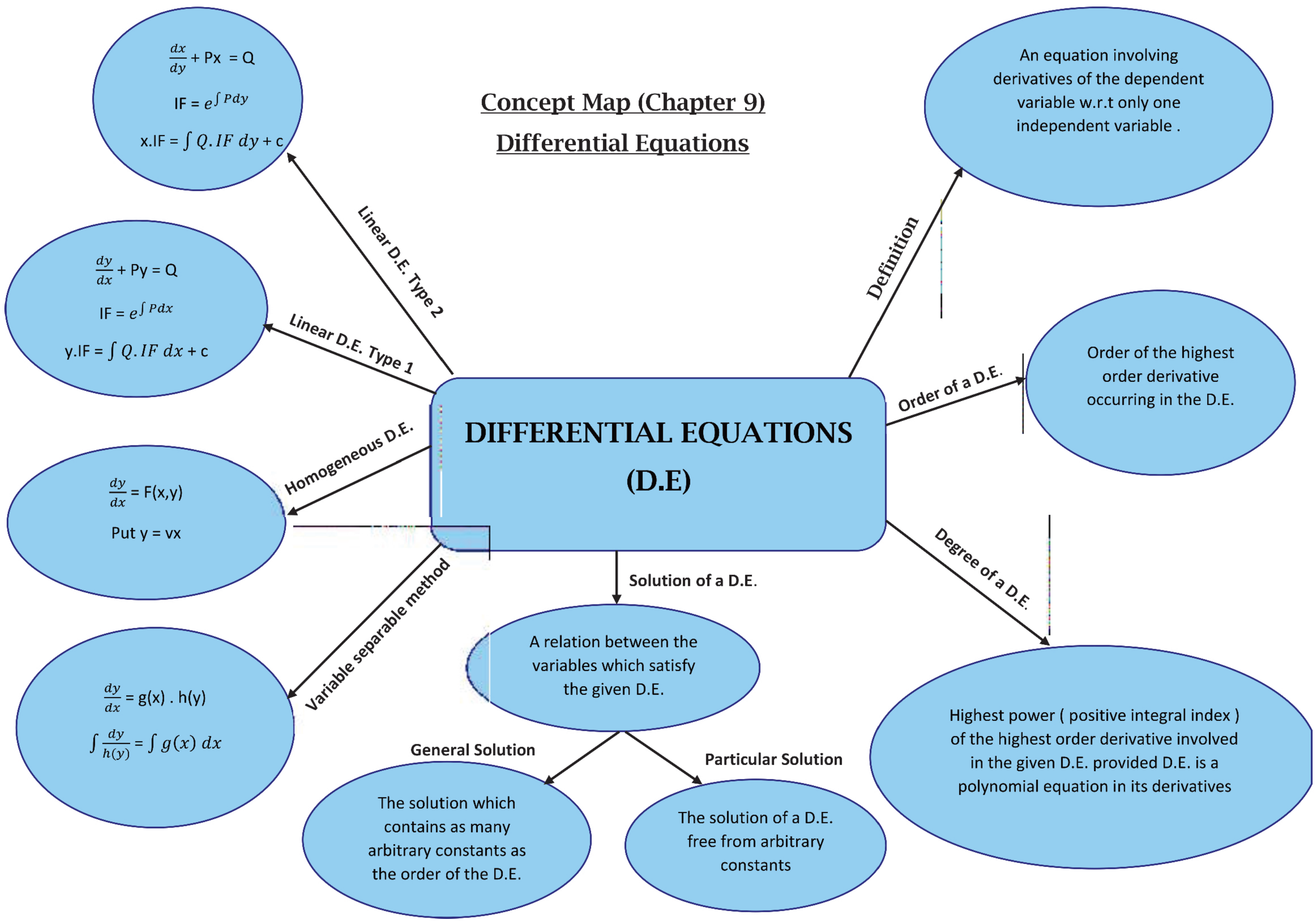
If the curves $y_1 = f(x)$ & $y_2 = g(x)$ intersect at two points $A(x = a)$ and $B(x = b)$, then the area between the curves is given by



$$Area = \int_a^b (y_1 - y_2) dx = \int_a^b (f(x) - g(x))$$

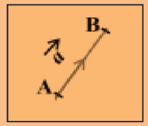
Concept Map (Chapter 9)

Differential Equations



CONCEPT – MAP (CHAPTER – 10)
VECTORS

A quantity that has magnitude as well as direction is called a vector.



A directed line segment is a vector denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

If a point P in space, having coordinates (x, y, z) with respect to the origin $O(0, 0, 0)$. Then, the vector $\overrightarrow{OP}$ having O and P as its initial & terminal points, respectively, is called the position vector of the point P with respect to O.

Using distance formula, the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

The angles made by $\overrightarrow{OP}$ with positive direction of x, y & z-axes (say α, β & γ respectively) are called *its direction* angles, and the cosine value of these angles i.e. $\cos\alpha, \cos\beta$ & $\cos\gamma$ are called direction cosines of $\overrightarrow{OP}$ *denoted by l, m & n respectively.*

TYPES OF VECTORS

- *A zero vector is a vector when the magnitude of the vector is zero and the starting point of the vector coincides with the terminal point.*
- *A vector which has a magnitude of unit length is called a unit vector.*
- *Two or more vectors which have the same starting point are called co-initial vectors.*
- *Two vectors are collinear if they are parallel to the same line irrespective of their magnitudes and direction.*
- *Two or more vectors are said to be equal when their magnitude is equal and also their direction is the same.*
- *Negative of a Vector: If two vectors are the same in magnitude but exactly opposite in direction then both the vectors are negative of each other.*

VECTORS

SCALAR(DOT) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ, *then* scalar product is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \le \theta \le \pi$

VECTOR(CROSS) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ, *then* vector product is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$, *where* $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ *and* $\vec{b}$.

Vector Joining Two Points

Let $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ be any two points in the space, then $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ &

$\overrightarrow{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

SECTION FORMULAE

The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio m : n

(i) Internally is given by $\dfrac{m\vec{b} + n\vec{a}}{m+n}$

(ii) Externally is given by $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

The position vector of middle point of PQ is given by $\dfrac{\vec{a} + \vec{b}}{2}$.

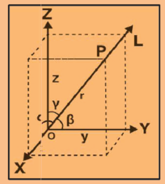
CONCEPT – MAP (CHAPTER – 11)
THREE DIMENSIONAL GEOMETRY

THREE DIMENSIONAL GEOMETRY

DIRECTION COSINES OF A LINE (DC'S)

The direction cosines are denoted by l, m, n.

Thus, $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$



- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
- $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

DIRECTION RATIO'S OF A LINE (DR's)

Any three numbers a, b and c proportional to the direction cosines l, m and n respectively are called direction Ratios of the line.

- The Direction ratios of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$.
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

EQUATION OF A LINE

- Equation of a line through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$

VECTOR FORM : $\vec{r} = \vec{a} + \lambda\vec{b}$

CARTESIAN FORM : $\frac{x-x_1}{p} = \frac{y-y_1}{q} = \frac{z-z_1}{s}$

Where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$,

$\vec{b} = p\hat{i} + q\hat{j} + s\hat{k}$

NOTE : $< p, q, s >$ are d.r.'s of the line

SHORTEST DISTANCE BETWEEN TWO SKEW – LINES

Let the lines are $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ & $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, then

$$S.D. = \left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

DISTANCE BETWEEN TWO PARALLEL – LINES

Let the lines are $\vec{r} = \vec{a_1} + \lambda\vec{b}$ & $\vec{r} = \vec{a_2} + \mu\vec{b}$, then

$$S.D. = \left|\frac{(\vec{a_2} - \vec{a_1}) \times \vec{b}}{|\vec{b}|}\right|$$

EQUATION OF A LINE

- Equation of a line through passing through two given point with position vector $\vec{a}$ and $\vec{b}$

VECTOR FORM : $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

CARTESIAN FORM : $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

Where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

ANGLE BETWEEN TWO LINES

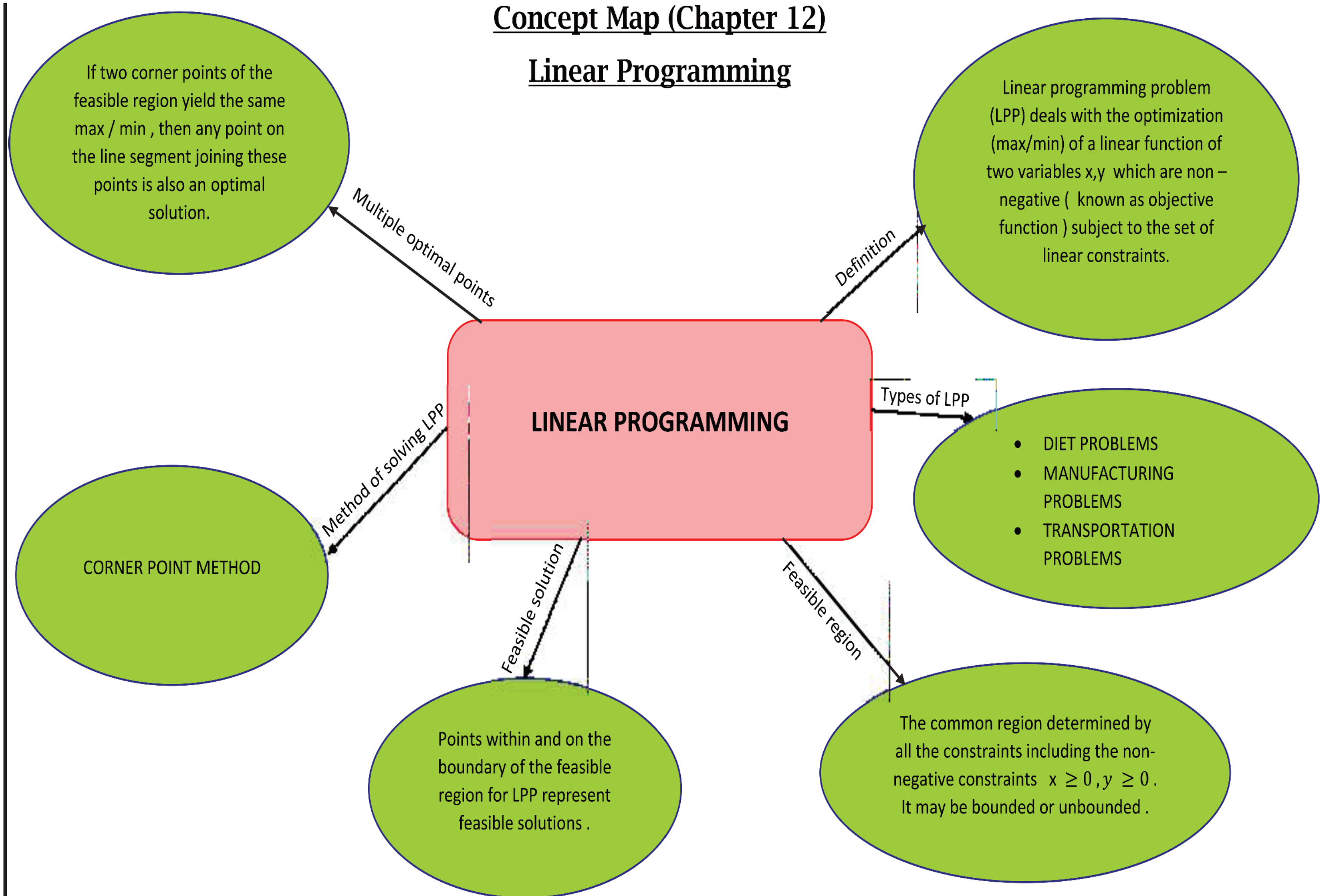
$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ & $L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right| = |l_1l_2 + m_1m_2 + n_1n_2|$$

- If two lines are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two lines are parallel then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

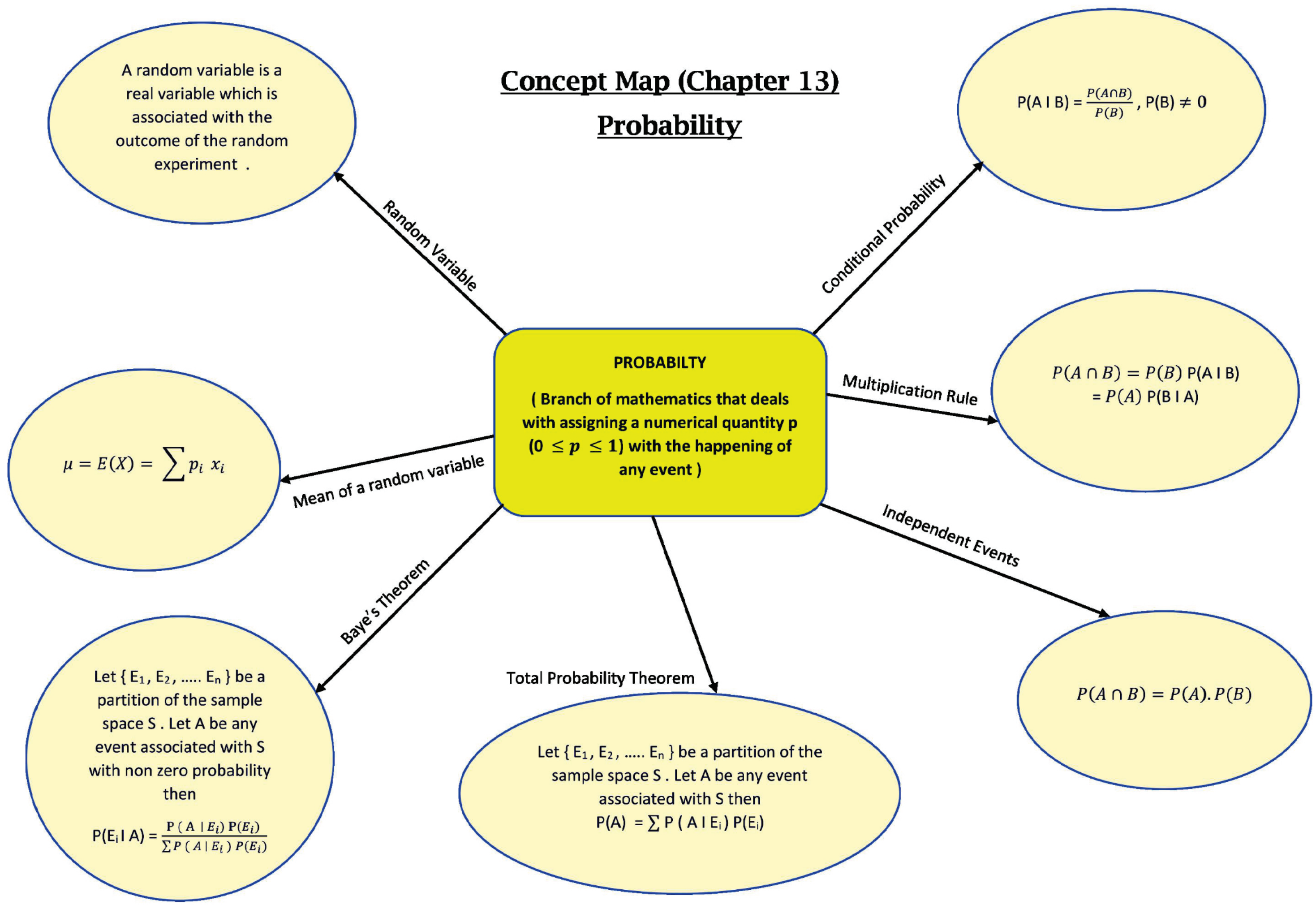
Concept Map (Chapter 12)

Linear Programming



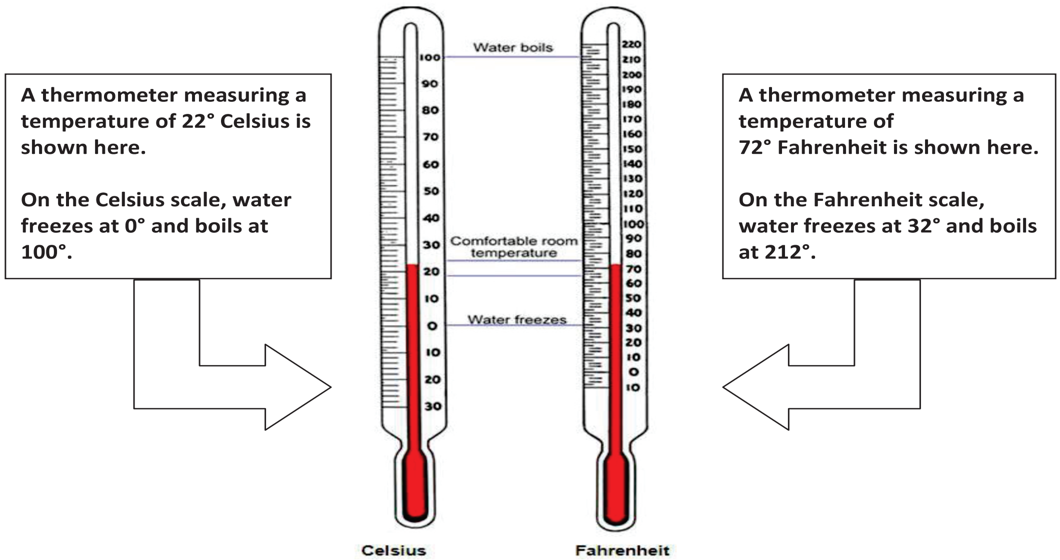
Concept Map (Chapter 13)

Probability



CHAPTER-1

RELATIONS AND FUNCTIONS



By looking at the the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenhelt measurement or vice versa.

For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process. Using the function

$F = f(C) = \frac{9}{5}C + 32$, any temperature in Celsius can be converted into Fahrenheit scale.

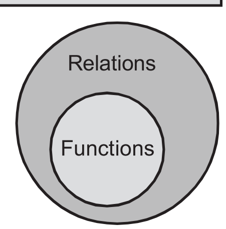
TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23
Types of relations: reflexive, symmetric, transitive and equivalence relations.
One to one and onto functions

A relation in a set A is a subset of A × A.

Thus, R is a relation in a set A = $R \subseteq A \times A$

If (a, b) ε R then we say that a is related to b and write, a R b

If (a, b) $\notin$ in K then we say that a is not related to b and write, a $\not R$ b.



If number of elements in set A and set B are p and q respectively, Means $n(A) = p$, $n(B) = q$, then

No. of Relation af A × A = 2^{p^2}

No. of Relation of B × B = 2^{q^2}

No. of Relation of A × B = No. of Relation of B × A = 2^{pq}

No. of **NON EMPTY** Relation of A × A = $(2^{p^2} - 1)$,

No. of **NON EMPTY** Relation of B × B = $(2^{q^2} - 1)$.

No. of **NON-EMPTY** Relation of A × B = No. of Relation of B × A = $(2^{pq} - 1)$

Q.1 If A = {*a, b, c*} and B = {1, 2} find the number of Relation R on (i) A × A (ii) B × B (iii) A × B

Ans. As $n(A) = 3$, $n(B) = 2$, so

No. of Relation R on A × A = $2^{3 \times 3} = 2^9 = 512$

No. of Relation R on B × B = $2^{2 \times 2} = 2^4 = 16$

No. of Relation R on A × 8 = $2^{3 \times 2} = 2^8 = 64$

Q.2 A = {*d*, *o*, *e*} and B = {22, 23} find the number of Non-empty Relation R on (i) A × A (ii) B × B

Ans. As $n(A) = 3$, $n(B) = 2$, so

No. of Relation Non-empty relations R on A × A = $2^{3 \times 3} - 1 = 2^9 - 1 = 511$

No. of Relation Non-empty R on B × B = $2^{2 \times 2} - 1 = 2^4 - 1 = 15$

Different types of relations

- **Empty Relation Or Void Relation**

  A relation R in a set A is called an empty relation, if no element of A is related to any element of A and we denote such a relation by

  Example: Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(*a, b*): $a + b = 20$}.

- **Universal Relation**

  A relation R in a set A is called an universal relation, if each element of A is related to every element of A.

  Example: Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(*a, b*): $a + b > 0$}.

- **Identity Relation**

  A relation R in a set A is called an identity relation, where R = {(*a, a*), $a \in$ A}.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by R = {(1, 1), (2, 2), (3, 3), (4, 4)}.

- **Reflexive Relation**

  A relation R in a set A is called a Reflexive relation, if $(a, a) \in$ R, for all $a \in$ A.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3), (4, 4)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1)}.

- **Symmetric Relation**

  A relation R in a set A is called a symmetric relation, if $(a, b) \in$ R, then $(b, a) \in$ R for all $a, b \in$ A.

  Example : Let A = {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3)}.

  R = {(1, 2), (2, 1), (3, 3)}.

  R = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (1, 3), (3, 1), (3, 2)}.

- **Transitive Relation**

  A relation R in a set A is called a transitive relation,

  if $(a, b) \in$ R and $(b, c) \in$ R then $(a, c) \in$ R for all $a, b, c \in$ A

  Or

  $(a, b) \in$ R and $(b, c) \notin$ R for all $a, b, c \in$ A

  Example : Let {1, 2, 3, 4} and let R be a relation in A, given by

  R = {(1, 1), (2, 2), (3, 3)}. (According to second condition)

  R = {(1, 2), (2, 1), (1, 1), (2, 2)}. (According to first condition)

  R = {(2, 3), (1, 3), (3, 1), (3, 2), (3, 3), (2, 2), (1, 1)}.

- **Equivalence Relation**

  A relation R in a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Illustration:

Let A be the set of all integers and let R be a relation in A, defined by R = {a, b}: $a = b$}, Prove that R is Equivalence Relation.

Solution: Reflexivity : Let R be reflexive $\Rightarrow (a, a) \in$ R $\forall\ \alpha \in$ A

$\Rightarrow$ $a = a$, which is true

Thus, R is Reflexive Relation.

Symmetricity : Let $(a, b) \in$ R $\forall\ a, b \in$ A

$\Rightarrow$ $a = b$

$\Rightarrow$ $b = a$

so $(b, a) \in R$. Thus R is symmetric Relation.

Transitivity : Let $(a, b) \in$ R and $(b, c) \in R\ \forall\ a, b, c \in$ A

$\Rightarrow\quad a = b$ and $b = c$

$\Rightarrow\quad a = b = c$

$\Rightarrow\quad a = c$

so $(a, c) \in$ R. Thus R is transitive Relation.

As, R is reflexive, Symmetric and transitive Relation

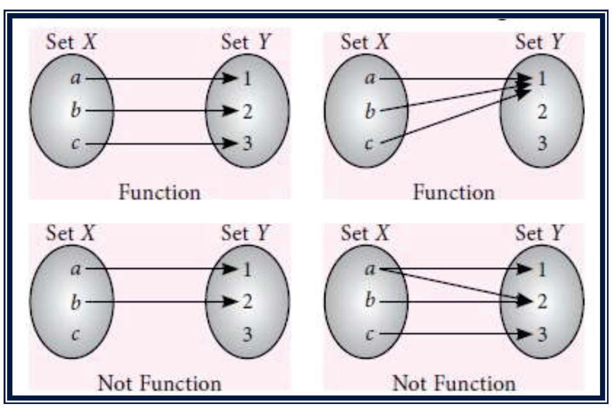
$\therefore$ R is an Equivalence Relation

FUNCTIONS

Functions can be easily defined with the help of concept mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be '*f*' then mathematically we write $f : x \rightarrow y$.

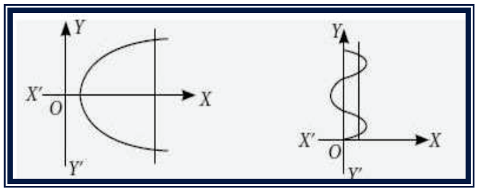
where $y = f(x)$, $x \in X$ and $y \in Y$. We say that '*y*' is the imagesof '*x*' under *f* (or x is the pre image of *y*).

- A mapping $f : X \rightarrow Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.
- Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X.
- Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from X and Y) eg.

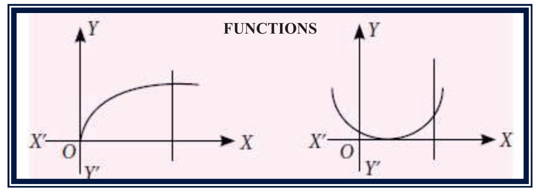


Testing for a function by Vertical line Test

A relation $f : A \rightarrow B$ is a function or not, it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the give curve at more thatn one point then the given relation is not a function and when this vertical line means line parallel to Y-asix cuts the curve at only one point then it is a function. Following figures represents which si not a function and which is a function.



NOT FUNCTION NOT A FUNCTION



Number of Functions
Let X and Y be two finite sets having m and n elements respectively. Thus each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

Real valued function: if R, be the set of real numbers and A, B are subsets of R, then the function f : A $\rightarrow$ B is called a real function or real valued functions.

Domain, Co-Domain And Range of Function

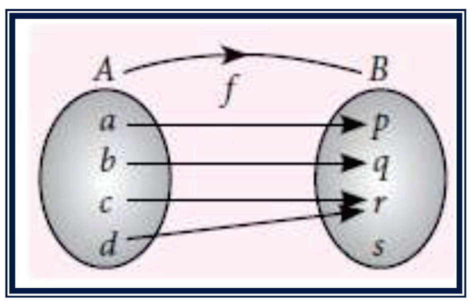
If a function *f* is defined from a set A to set B then (if : A $\rightarrow$ B) set A is called the domain of f and set B is called the co-domian of *f*.

The set of all *f*-images of the elements of A is called the range of *f*.

In other words, we can say

Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of , all possible values of $f(x)$.



From the figure we observe that

Domain = A = $\{a, b, c, d\}$ Range = $\{p, q, r\}$, Co-Domain = $\{p, q, r, s\}$ = B

EQUAL FUNCTION

Two function *f* and *g* are said to be equal functions, if and only if

(i) Domain of f = Domain of g

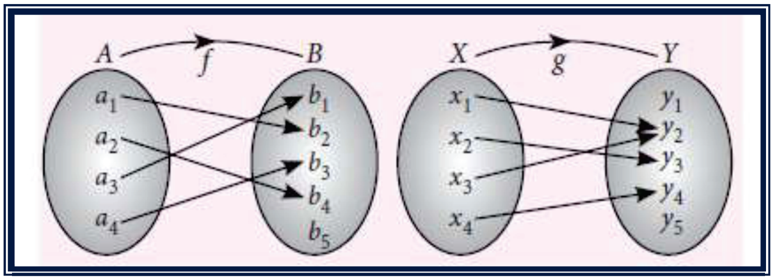
(ii) Co-domain of *f* = Co-domain of g

(iii) $f(x) = g(x)$ for all $x \in$ their common domain

TYPES OF FUNCTION

One-one function (injection): A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B.

e.g. Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams.



Clearly, $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function g.

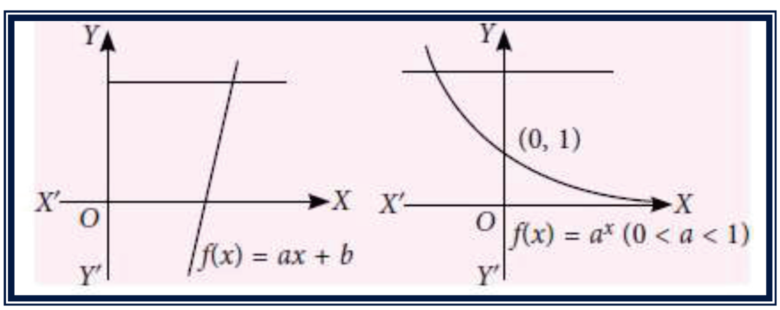
Method to check the injectivity (One-One) of a function

(i) Take two arbitrary elements x, y (say) in the domain of f.

(ii) Solve $f(x) = f(y)$. If $f(x) = f(y)$ give $x = y$ only, then $f : A \to B$ is a one-one function (or an injection). Otherwise not.

If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

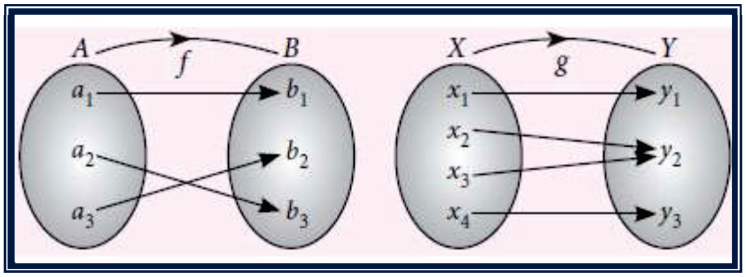
If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one. (Strictly increasing or Strictly Decreasing Function). E.g.



Number of one-one functions (injections) : If A and B are finite sets having m and n elements respectively, then number of one-one functions from A and B = ${}^nP_m$ is $n \geq m$ and 0 if $n < m$.

If $f(x)$ is not one-one function, then its Many-one function.

Onto function (surjection) : A function $f : A \to B$ is onto if each element of B has its pre-image in A. In other words, Range of f = Co-domain of f. *e.g.* The following arrow-diagram shows onto function.

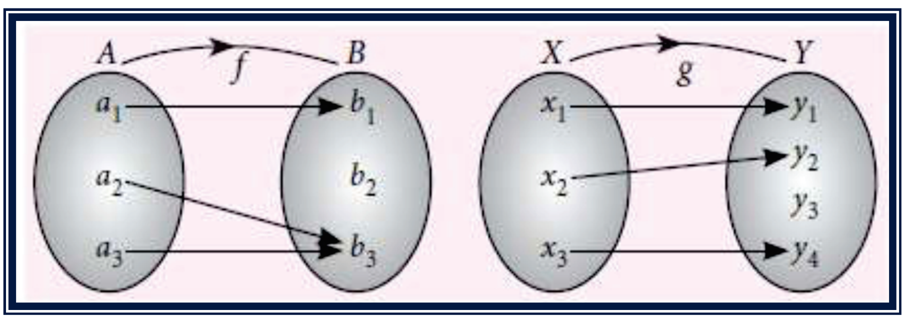


Number of onto function (surjection): If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r} \ {}^{n}C_r$

$. \, r^m$

Into function: A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f: A \to B$ is an into function if it is not an onto function e.g, The following arrow diagram shows into function.



Method to find onto or into function:

(i) Solve $f(x) = y$ by taking x as a function of y i.e., $g(y)$(say).

(i) Now if $g(y)$ is defined for each $y \,\varepsilon$ co-domain and $g(y) \,\varepsilon$ domain then f(x) is onto and if any one of the above requirements is not fulfilled, then fix) is into.

One-one onto function (bijection) : A function $f: A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f: A \to$ Bis a bijection if

(i) It is one-one ie., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in$ A.

(ii) It is onto i.e., for all $y \in$ B, there exists $x \in$ A such that $f(x) = y$.

Clearly, f is a bijection since it is both injective as well as surjective.

Illustration :

Let $f: R \to R$ be defined as $f(x) = 7x - 5$, then show that function is one-one and onto Both.

Solution : Let $f(x) = f(y)\ \forall\ x, y \in$ R

$\Rightarrow \quad 7x - 5 = 7y - 5$

$\Rightarrow$ $x = y$, so **$f(x)$ is one-one function**

Now, As $f(x) = 7x - 5$, is a polynomial function.

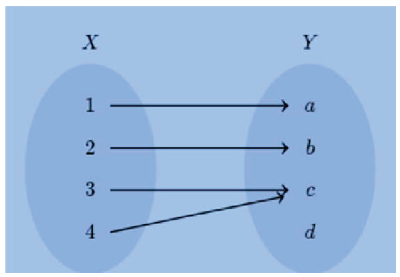
so it is defined everywhere. Thus, Range = R

As, Range = co-domain, so **f is onto function.**

Alternative method : Graph of $f(x)$ is a line which is strictly increasing for all values of x, so its on-one function and Range of $f(x)$ is R which is equal to R so onto funclion.

ILLUSTRATION:

If $f : X \to Y$ is defined, then show that f is neither one-one nor onto function.



Solution : As for elements 3 and 4 from set X we have same image c in set Y, so f is not one-one function.

Further element d has no pre -image in set X,

so f is not onto function

ILLUSTRATION:

Prove that the fiunction $f : N \to N$, defined by $f(x) = x^2 + x + 2022$ is one-one.

SOLUTION : APPROACH-I

Let $f(x_1) = f(x_1)\ \forall\ x_1, x_2 \in N \Rightarrow = x_1^2 + x_1 + \cancel{2022} = x_2^2 + x_2 + \cancel{2022}$

$\Rightarrow$ $x_1^2 + x_1 = x_2^2 + x_2$

$\Rightarrow$ $(x_1^2 - x_2^2) + (x_1 - x_2) = 0$

$\Rightarrow$ $(x_1 - x_2) + (x_1 + x_2 + 1) = 0$

Thus, $(x_1 - x_2) = 0$ as $(x_1 + x_2 + 1) \neq 0\ \forall\ x_1, x_2 \in N$

so, f is ONE-ONE function

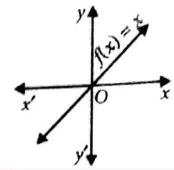
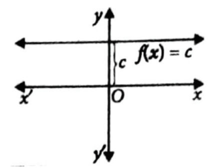
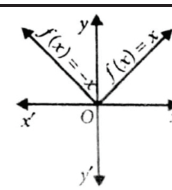
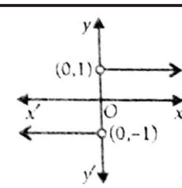
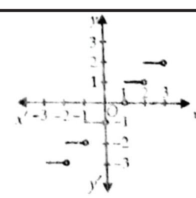
APPROACH-II

$f(x) = x^2 + x + 2022 \Rightarrow f'(x) = 2x + 1$

As, $x \in N$ so, $2x + 1 > 0 \Rightarrow f'(x) = 0$ (Strictly Increasing function)

so, f is ONE-ONE function

Type of Functions

Name of Function	Definition	Domain	Range	Graph
1. Identify Function	The function $f : R \to R$ defined by $f(x) = x \ \forall \ x \in R$	R	R	
2. Constant Function	The function $f : R \to R$ defined by $f(x) = c \ \forall \ x \in R$	R	(c)	
3. Polynomial Function	The function $f : R \to R$ defined by $f(x) = p_0 + p_1x + p_2x^2 + ... + p_nx^n$, where $n \in N$ and $p_0, p_1, p_2 ..., p_n \in R \ \forall \ x \in R$			
4. Rational Function	The function f defined by $f(x) = \frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomial functions, $Q(x) \neq 0$			
5. Modulus Function	The function $f : R \to R$ defined by $f(x) = \|x\| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \ \forall \ x \in R$	R	$[0, \infty)$	
6. Signum Function	The function $f : R \to R$ defined by $f(x) = \begin{cases} \frac{\|x\|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ 0, & x = 0 \end{cases}$	R	{–1,0,1}	
7. Greatest Integer Function	The function $f : R \to R$ defined by $f(x) = \|x\| = \begin{cases} x, x \in Z \\ \text{integer less than} \\ \textit{equal to } x, x \notin Z \end{cases}$	R	Z	
8. Linear Function	The function $f : R \to R$ defined by $f(x) = mx + c$, $x \in R$ where m and c are constants	R	R	

ONE–MARK QUESTIONS

1. Consider the set A = {1, 2, 3}, then write smallest equivalence relation on A.
2. Consider the set A containing n elements then, write the total number of injective functions from A onto itself.
3. Let Z be the set of integers and R be the relation defined in Z such that a R b if $(a - b)$ is divisible by 3, then R partitions the set Z into how many Pairwise disjoint subsets.
4. Let the relation R be defined in N by a R b if $2a + 3b = 30$. Find R.
5. Let R be the equivalence relation in the set Z of integers given by R = {(a, b) : 2 divides a – b}, then Find the equivalence class [0].
6. Let A = {1, 2, 3, ..,n} and B = {a, b}. Find the number of surjections from A to B.

   Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is sister of b, Then check R is Transitive or not.
8. Find the maximum number of equivalence relations on the set A = {1, 2, 3}.
9. If $f(x) = x^2 - \frac{1}{x^2}$, then find $f\left(\frac{1}{x^2}\right) + f(x^2)$.
10. Show that the function f : R $\rightarrow$ R defined by $f(x) = x^2$ is not Injective.
11. Show that the function f: N $\rightarrow$ N given by $f(x) = 3x$ is not Surjective.
12. Find the largest Equivalence Relation on A = {a, b, c}.
14. Set A has 3 elements and the set B has 4 elements. Find the number of injective mappings that can be defined from A to B.
15. Find the maximum number of reflexive relation on the set A = {a, b}.
16. If A is the set of students of a school then write, which of following relations are Universal, Empty or neither of the two.

    R_1 = {(a, b) :a, b are ages of students and $|a - b| > 0$}

    R_2 = {(a, b) : a, b are weights of students, and $|a - b| < 0$}

    R_3 = {(a, b) : a, b are students studying in same class}
17. If f : A $\rightarrow$ B is Bijective function such that n(A) = 10, then find n (B).
18. If f : R $\rightarrow$ B given by $f\{x) = \sin x$ is onto function, then write set B.
19. Let A = {a, b, c}. How many relation can be defined on A × A? How many of these are reflexive?
20. Let A = {1, 3, 5), then find the number of equivalence relations in A containing (1,3).

TWO MARKS QUESTIONS

21. If A = {a, b, c, d} and f = {(a, b), (b, d), (c, a), (d, c)}, show that f is one-one from A to A.
22. Show that the relation R on defined as R = {(a, b) : $a \leq b^3$} is not transitive.

23. If the function $f : R - \{1, -1\} \rightarrow A$ defined by $f(x) = \frac{x^2}{1 - x^2}$, is Surjective, then find A.

24. Give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on A.

25. How many reflexive relations are possible in a set A whose (A) = 4. Also find How many symmetric relations are possible on a set B whose $n(B) = 3$.

26. Let W denote the set of words in the English dictionary. Define the relation R by R $\{(x, y)$ ε W × W such that x and y have at least one letter in common). Show that this relation R is reflexive and symmetric, but not transitive.

27. Show that the relation R in the set of all real numbers, defined as $R = \{(a, b): a \leq b^2\}$ is neither reflexive Nor symmetric.

28. Consider a function $f : R_+ \rightarrow (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R+ is the set of all positive real numbers. Show that function is one-one and onto both.

29. Let L be the set of all lines in a plane. A relation R in Lis given by R $\{(L_1, L_2)$: L_1 and L_2 intersect at exactly one point, $L_1, L_2 \varepsilon L\}$, then show that the relation R is symmetric Only.

30. A relation R on set of Natural numbers is given by R = $\{(x, y)$: xy is a square of an integer) is Transitive.

THREE MARKS QUESTIONS

31. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

    (i) $\{(x, y) : x$ is a person, y is the mother of $x\}$.

    (ii) $\{(a, b)$: a is a person, b is an ancestor of $a\}$.

32. Show that the function $f : R \rightarrow R$ defined by $f(x) = \frac{x^2}{x^2 + 1}$; $\forall\ x \in$ R, is neither one-one nor onto.

33. Let R be the set of real numbers and $f : R \rightarrow R$ be the function defined by $f(x) = 4x + 5$. Show that f is One-one and onto both.

34. Show that the relation R in the set A = {3, 4, 5, 6, 7)} given by R = $\{(a, b) : |a - b|$ is divisible by 2} is an equivalence relation. Show that all the elements of {3, 5, 7} are related to each other and all the ements of {4, 6} are related to each other, but no element of {3, 5, 7} is related to any element {4, 6}.

35. Check whether the relation R in the set Z of integers defined as R = $((a, b) : a + b$ is "divisible by 2"} is reflexive, symmetric, transitive or Equivalence.

36. Show that that following Relations R are equivalence relation in A.

    (a) Let A be the set of all triangles in a plane and let R be a relation in A, defined by $R = \{(T_1, T_2) : T_1$, is congruent $T_2\}$

    (b) Let A be the set of all triangles in a plane and let R be a relation in A, defined by $R = \{(T_1, T_2) : T_1$, is similar T_2,}

(c) Let A be the set of all lines in xy-plane and let R be a relation in A, defined by

R = {(L_1, L_2,) : L_1, is parallel to L_2}

(d) Let A be the set of all integers and let R be a relation in A, defined by

R = {(a, b) : (a – b) is even}

(e) Let A be the set of all integers and let R be a relation in A, defined by

R = {(a, b) : $|a - b|$ is a multiple of 2}

(f) Let A be the set of all integers and let R be a relation in A, defined by

R = {(a, b) : $|a - b|$ is a divisible by 3}

37. Check whether the following Relations are Reflexive, Symmetric or Transitive.

(a) Let A be the set of all lines in *xy*-plane and let R be a relation in A, defined by

R = {(L_1, L_2) : L_1 is perpendicular to L_2}

(b) Let A be the set of all real numbers and let R be a relation in A defined by

R = {(a, b): $a \leq b$}

(c) Let A be the set of all real numbers and let R be a relation in A defined by

R = {(a, b): $a \leq b^2$}

(d) Let A be the set of all real numbers and let R be a relation in A defined by

R = {(a, b) : $a \leq b^3$}

(e) Let A be the set of all natural numbers and let R be a relation in A defined by

R = (a, b) : a is a factor b}

OR

R {(a.b): b is divisible by a}

(f) Let A be the set of all real numbers and let R be a relation in A defined by

R {(a.b): (1+ ab) > 0}

38. Let S be the set of all real numbers. Show that the relation R = {a, b): $a^2 + b^2 = 1$} is symmetric but neither reflexive nor transitive.

39. Check whether relation R defined in R as R = {a, b): $a^2 - 4ab + 3b^2 = 0$, $a, b \in$ R} is reflexive, symmetric and transitive.

40. Show that the function $f : (-\infty, 0) \to (-1, 0)$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in (-\infty, 0)$ is one-one and onto.

FIVE MARKS QUESTIONS

41. For real numbers x and y, define $x\ R\ y$ if and only if $x - y + \sqrt{2}$ is an irational number. Then check the reflexivity, Symmetricity and Transitivity of the relation R.

42. Determine whether the relation R defined on the set of all real numbers as

R = {(a, b) : a, b ε R and a – b + $\sqrt{3}$ ε S}

(Where S is the set of ail irrational Numbers) is reflexive, symmetric or transitive.

43. Let N be the set of ali natural numbers and let R be a relation on N × N, defined by Show that R is an equivalence relation.

(i) (a, b) R $(c, d) \Leftrightarrow a + d = b + c$

(ii) (a, b) R $(c, d) \Leftrightarrow ad = bc$

(iii) (a, b) R $(c, d) \Leftrightarrow \frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$

(iv) (a, b) R $(c, d) \Leftrightarrow ad\,(b + c) = bc\,(a + d)$

44. Let $A = R - \{1\}$, $f : A \to A$ is a mapping defined by $f(x) = \frac{x-2}{x-1}$, show that f is one-one and onto.

45. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to S$, where S is the range of f, is One-One and Onto Function.

CASE STUDIES

A. A person without family is not complete in this world because family is an integral part of all of us Human deings are considered as the social animals living in group called as family. Family plays many important roles throughout the life.

Mr. D.N. Sharma is an Honest person who is living happily with his family. He has a son Vidya and a Daughter Madhulika. Mr. Vidya has 2 sons Tarun and Gajender and a daughter Suman while Mrs. Madhulika has 2 sons Shashank and Pradeep and 2 daughters Sweety and Anju. They all Lived together and everyone shares equal responsibilities within the family. Every member of the family emotionally attaches to each other in their happiness and sadness. They help each other in their bad times which give the feeling of security.

A family provides love, warmth and security to its all members throughout the life which makes it a complete family. A good and healthy family makes a good society and ultimately a good society involves in making a good country.

On the basis of above information, answer the following questions:

Consider Relation R in the set A of members of Mr. D. N. Sharma and his family at a particular time

(i) If R = {(x, y) : x and y live in the same locality), then R is

(a) Reflexive only

(b) Reflexive and Symmetric

(c) Retlexive and Transitive

(d) Equivalence Relation

(ii) R = {(x, y) : x is exactly 7 cm taller than y}, then R is

(a) Reflexive only

(b) Symmetric only

(c) Transitive only

(d) Neither Reflexive, symmetric nor Transitive

(iii) R = {(x, y) : x is wife of y}, then R is

(a) Reflexive only

(b) Symmetric only

(c) Transitive only

(d) Neither Reflexive, symmetric nor Transitive

(iv) R = {(x, y): x is father of y}, then R is

(a) Equivalence Relation

(b) Symmetric only

(c) Transitive only

(d) Neither Reflexive, symmetric nor Transitive

(v) R = {(x, y): x is Brother of y}, then R is

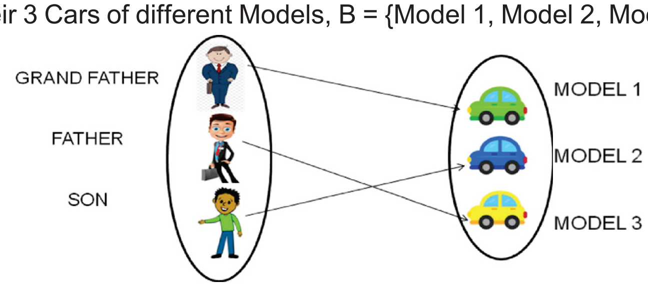
(a) Reflexive only

(b) Symmetric only

(c) Transitive only

(d) Neither Reflexive, symmetric nor Transitive

B. Let A be the Set of Male members of a Family, A = (Grand father, Father, Son) and B be the set of their 3 Cars of different Models, B = {Model 1, Model 2, Model 3}



On the basis of The above Information, answer the following questions:

(i) How many Relations are possible on A × B?

(a) 3 (b) 9

(c) 8 (d) 512

(ii) How many Functions are possible on A × B?

(a) 3 (b) 9

(c) 27 (d) None of these

(ii) How many One-one Functions (Injective) are possible on A × B?

(a) 3 (b) 6

(c) 9 (d) 12

(iv) How many Onto Functions (surjective) are possible on A × B?

(a) 6 (b) 9

(c) 27 (d) 81

(v) How many Bijective functions are possible on A × B?

(a) 1 (b) 3

(c) 6 (d) 9

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE:

1. Consider the set A = {1. 2, 3} and R be the smallest equivalence relation on A.then R =

(a) {(1.1)} (b) {(1,1), (2,2)}

(c) {(1,1),(2,2),(3,3)} (d) ϕ

2. Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is

(a) 2^n (b) n

(c) n (d) $n!$

3. The total number of injective nappingsfrom a set with m elements to a set with n elements, $m \leq n$ is

(a) $n!$ (b) n^m

(c) m^n (d) $\dfrac{n!}{(n-m)!}$

4. The number of injections possible from A = {1,3,5,6} to B = {2,8,11} is

(a) 12 (b) 22

(c) 3 (d) 0

5. The number of one-one functions that can defined from

   A = {4,8,12,16} to B is 5040, then $n(B)$ =

   (a) 7 (b) 8

   (c) 9 (d) 10

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT CHOOSE THE CORRECT ALTERNATIVE.

1. A relation R in a set A is called .... if $(a_1, a_2) \in$ R implies $(a_2, a_1) \in$ R, for all a_1, a2 $\in$ A.

   (a) Reflexive (b) Simmetric

   (c) Transitive (d) Equivalence

2. Let $f : R - \{0\} \rightarrow R - \{0\}$ be defined by $f(x) = \frac{1}{x} \forall x \in$ R. Then f is

   (a) One-One (b) Many-One

   (c) Not defined (d) None of these

3. Let P = $\{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then P is

   (a) Re flexive (b) Symmetric

   (c) Transitive (d) Equivalence

4. The function $f : R \rightarrow R$ defined by $f(x) = [x]$, where [.] is greatest integer function is

   (a) One-One (b) Many -One

   (c) Onto (d) None of these

5. The number of bijective functions (One-one and onto both) from set A to itself when A contains 2022 elements is

   (a) 2022 (B) 2022!

   (C) 2022^2 (D) 2022^{2022}

ANSWER

One Mark Questions

1. {(1,1), (2,2), (3,3)} 2. $n!$ 3. Three
4. {(3,8), (6, 6), (9,4), (12,2)} 5. [0] = {0, ±2, ±4, ±6...}
6. $(2^n - 2)$ 7. Yes, Transitive
8. Five 9. Zero 13. A × A
14. 24 15. Four
16. R_1 : Universal relation, R_2 : Empty relation, R_3 Neither universal Nor empty
17. n(B) = 10 18. B [–1,1]
19. No. of Relation = 512 No, of Relexive Relations = 2^6 = 64
20. Two

Two Mark Questions

23. A = R – [–1, 0]

24. Reflexive Relations = 4096 Symmetric Relation = 64

Three Mark Questions

31. (a) Yes it's function, Not Injective but Surjective (b) No, its not a function

35. EQUIVALENCE RELATION

37. (a) Symmetric (b) Reflexive and Transitive
    (c) Neither Reflexive, Symmetric nor Transitive
    (d) Neither Reflexive, Symmetric nor Transitive
    (e) Reflexive and Transitive
    (f) Reflextive and Symmetric

39. Reflexive only

Four/Five Mark Questions

41. Reflexive only 42. Reflexive only

CASE STUDIES BASED QUESTION

A. (i) option (d)	A. (ii) option (d)
A. (iii) option (c)	A. (iv) option (d)
A. (v) option (c)	A. (i) option (d)
A. (ii) option (c)	A. (iii) option (b)
A. (iv) option (a)	A. (v) option (c)

SELF ASSESSMENT-1

1. (c)	2. (d)	3. (d)	4. (d)	5. (d)

SELF ASSESSMENT-2

1. (b)	2. (a)	3. (b)	4. (b)	5. (b)

CHAPTER-2

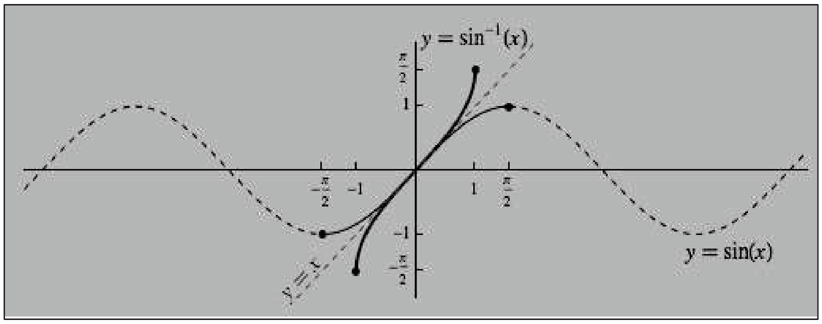
INVERSE TRIGONOMETRIC FUNCTIONS

An example of people using inverse trigonometric functions would be builders such as construction workers, architects, and many others.

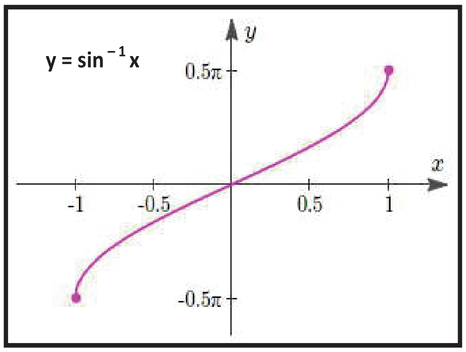
An example of the use would be the creation of bike ramp. You will have to find the height and the length. Then find the angle by using the inverse of sine. Put the ength over the height to find the angle. Architects would have to calculate the angle of a bridge and the supports when drawing outlines. These calculations are then applied to find the safest angle. The workers would then uses these calculations to build the bridge.

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23

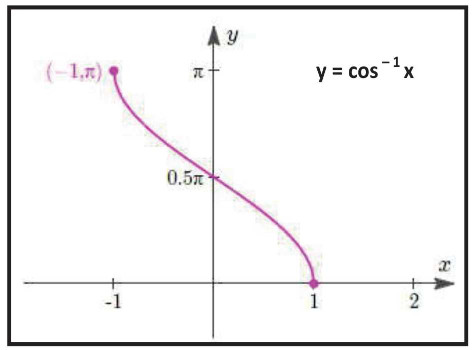
- Definition, range, domain, principal value branch.
- Graphs of inverse trigonometric functions.



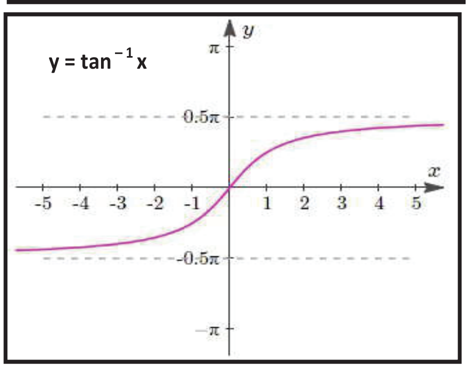
y = sin⁻¹(x)
π/2
1
−π/2
−1
1
π/2
−1
−π/2
y = x
y = sin(x)



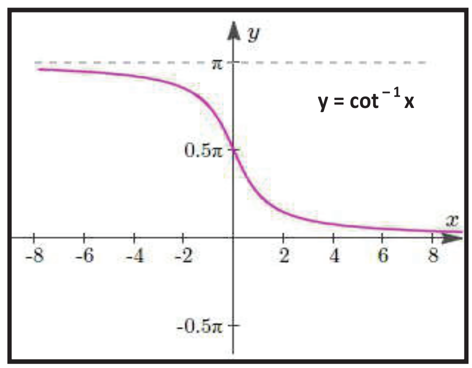
y = sin⁻¹x
y
0.5π
x
-1
-0.5
0.5
1
-0.5π



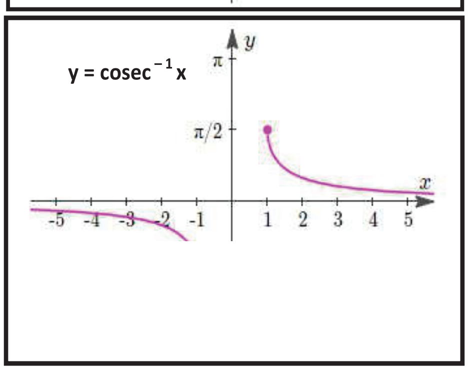
y
(−1,π)
π
y = cos⁻¹x
0.5π
x
-1
1
2



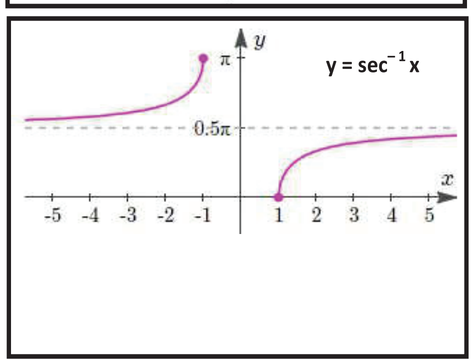
y = tan⁻¹x
y
π
0.5π
x
-5
-4
-3
-2
-1
1
2
3
4
5
-0.5π
−π



y
π
y = cot⁻¹x
0.5π
x
-8
-6
-4
-2
2
4
6
8
-0.5π



y
y = cosec⁻¹x
π
π/2
x
-5
-4
-3
-2
-1
1
2
3
4
5



y
π
y = sec⁻¹x
0.5π
x
-5
-4
-3
-2
-1
1
2
3
4
5

Function	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$
$y = \sec^{-1} x$	R – (–1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \text{cosec}^{-1} x$	R – (–1, 1)	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- when $x \in [-1, 1]$

  $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
- when $x \in [-1, 1]$

  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.
- when $x \in R - (-1, 1)$

  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$.

- $\sin^{-1} (\sin x) = x$, when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1} (\cos x) = x$, when $x \in [0, \pi]$
- $\tan^{-1} (\tan x) = x$, when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot^{-1} (\cot x) = x$, when $x \in (0, \pi)$
- $\text{cosec}^{-1} (\text{cosec } x) = x$, when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sec^{-1} (\sec x) = x$, when $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

- $\sin (\sin^{-1} x) = x$, when $x \in [-1, 1]$
- $\cos (\cos^{-1} x) = x$, when $x \in [-1, 1]$
- $\tan (\tan^{-1} x) = x$, when $x \in R$
- $\cot (\cot^{-1} x) = x$, when $x \in R$
- $\cot (\cos^{-1} x) = x$, when $x \in R$
- $\text{cosec} (\text{cosec}^{-1} x) = x$, when $x \in R - (-1, 1)$
- $\sec (\sec^{-1} x) = x$, when $x \in R - (-1, 1)$

- $\sin^{-1}(-x) = \sin^{-1}$, when $x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi \cos^{-1}(x)$, when $x \in [-1, 1]$
- $\tan^{-1}(-x) = \tan^{-1}(x)$, when $x \in R$
- $\cot^{-1}(-x) = \pi \cot^{-1}(x)$, when $x \in R$
- $\text{cosec}^{-1}(-x) = \text{cosec}^{-1}(x)$, when $x \in R\ (-1, 1)$
- $\sec^{-1}(-x) = \pi \sec^{-1}(x)$, when $x \in R\ (-1, 1)$

Illustration:

Find the principal value of $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$.

Solution: As, $\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$, $\boxed{\frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]}$

$\cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3}$, $\boxed{\frac{\pi}{3} \in [0, \pi]}$

so, $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

Illustration:

Find the principal value of $\sec^{-1}(2) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})$.

Solution: As, $\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right)$

$\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}) = -\tan^{-1}\left(\tan\frac{\pi}{3}\right) = -\frac{\pi}{3}$, $\boxed{-\frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

$\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3}) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

Illustration:

Find the range of the function $f(x) = \tan^{-1} x + \cot^{-1} x$.

Solution: As, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

so, $f(x) = \frac{\pi}{2}$ (A constant function)

Thus range of $f(x)$ is $\left\{\frac{\pi}{2}\right\}$.

Illustration:

If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then find the value of $\cos^{-1} x + \cos^{-1} y$.

Solution: As, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \Rightarrow \boxed{\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x}$

$\cos^{-1} x + \cos^{-1} y = \pi - (\sin^{-1} x + \sin^{-1} y) = \pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$

Illustration:

If $a \leq 2 \sin^{-1} x + \cos^{-1} x \leq b$, then find the value a and b.

Solution: We know that, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ and $\frac{-\pi}{2} \leq \sin^{-1} x \frac{\pi}{2}$,

$\Rightarrow \quad 0 \leq (\sin^{-1} x) + \frac{\pi}{2} \leq \pi$

$\Rightarrow \quad 0 \leq (\sin^{-1} x) + \sin^{-1} x + \cos^{-1} x \leq \pi$

$\Rightarrow \quad 0 \leq 2 \sin^{-1} x + \cos^{-1} x \leq \pi$, but given, $a \leq \sin^{-1}, x + \cos^{-1} x \leq b$

$\boxed{\text{Thus, } a = 0 \text{ and } b = \pi}$

Illustration:

If sin[$\cot^{-1}(1 + x)$] = cos[$\tan^{-1} x$], then find x.

Solution: As, sin[$\cot^{-1}(1 + x)$] = cos[$\tan^{-1} x$]

$$\Rightarrow \quad \sin\left[\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right] = \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$\Rightarrow \quad x^2 + 2x + 2 = 1 + x^2$

$\Rightarrow \quad 2x = -1 \Rightarrow \boxed{x = -0.5}$

Illustration:

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$.

Solution: Let, $\tan^{-1} x = A$, $\tan^{-1} y = B$, $\tan^{-1} z = C$

so, $A + B + C = \frac{\pi}{2} \Rightarrow A + B = \frac{\pi}{2} - C$

$$\tan(A + B) = \tan\left(\frac{\pi}{2} - C\right) = \cot C$$

$$\frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{1}{\tan C} \Rightarrow \frac{x+y}{1-xy} = \frac{1}{z}$$

$\Rightarrow \quad xz + yz = 1 - xy$

$\Rightarrow \quad \boxed{xz + yz + zx = 1}$

ONE MARK QUESTIONS

1. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$.

2. Find the principal value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.

3. Find the principal value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$.

4. If $\cos\left(\cos^{-1}\frac{1}{3} + \sin^{-1} x\right) = 0$, then find the value of x.

5. If $\sin\left(\sin^{-1}\frac{3}{5}+\cos^{-1}x\right)=1,$ then find the value of x.

6. Express $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}(x)$.

7. Find the domain of the function $\cos^{-1}(2x-1)$.

8. Find the domain of the function $f(x) = \sin^{-1}\sqrt{x-1}$.

9. Find the value of $\cot\left(\cos^{-1}\frac{7}{25}\right)$.

10. Find the minimum value of n for which $\tan^{-1}\frac{n}{\pi}>\frac{\pi}{4}, n \in N$.

11. Find the value of x, If $3\tan^{-1}x + \cot^{-1}x = \pi$.

12. Find the principal value in each of the following:

(a) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(b) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(c) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

(d) $\sin^{-1}\left(\sin\frac{2\pi}{5}\right)$

(e) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)+\cos^{-1}\left(\cos\frac{11\pi}{6}\right)$

13. If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$ then find the value of $\cot^{-1}x+\cot^{-1}y$.

14. Find the value of the expression $\sin[\cot^{-1}(\cos(\tan^{-1}1))]$.

15. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{2},$ then find the value of $\cot^{-1}x+\cot^{-1}y$.

16. Find the value of $\sin(2\sin^{-1}(0.6))$.

17. If $\tan^{-1}x=\frac{\pi}{10}$, for some $x \in R$, then find the value of $\cot^{-1}x$.

18. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then find the value of $(x + y + z + xyz)$.

19. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then find the value of $(x^3 + y^3 + z^3 - 3xyz)$.

20. Find x, such that $\cos^{-1}(x) + \cos^{-1}(x^2) = 0$.

21. Find x, if $\sin^{-1}\left(\frac{x}{5}\right)+\operatorname{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$

22. Find the range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$.

23. If the function $f(x) = \cot^{-1}\sqrt{x(x+3)}+\cos^{-1}\sqrt{x^2+3x+1}$ is defined on the set A, Find A.

24. If $\sum_{i=1}^{2022}\cos^{-1}x_i = 0,$ then find the value of $\sum_{i=1}^{2022} x_i$.

25. If $\sum_{i=1}^{2022}\sin^{-1}x_i = 1011\pi,$ then find the value of $\sum_{i=1}^{2022} x_i$.

TWO MARKS QUESTIONS

26. Match the following:

If $\cos^{-1}a + \cos^{-1}b = 2\pi$ and $\sin^{-1}c + \sin^{-1}d = \pi$ then

	Column 1		Column 2
A	abcd	P	0
B	$a^2 + b^2 + c^2 + d^2$	Q	1
C	$(d - a) + (c - d)$	R	2
D	$a^3 + b^3 + c^3 + d^3$	S	4

27. Find the value of $\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)\right]$

28. If $P = \tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3)$, then find the value of $(P^2 + P + 11)$.

29. If $P = \sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$, then find the value of $(P^2 - 2P)$.

30. Find the value of $\sin\left(\frac{1}{2}\cot^{-1}\left(\frac{3}{4}\right)\right)$. $\left[\text{Hint} : \sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}\right]$

31. Solve for x : $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

32. Find the value of x, such that $\sin^{-1}x = \frac{\pi}{6}+\cos^{-1}x$.

33. Find x, if $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{2}$

34. If $\tan^{-1}(\cot x) = 2x$, find x.

35. Solve for x : $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)+\sin^{-1}\left(\sin\frac{3\pi}{4}\right)=x$

THREE MARKS QUESTIONS

36. Find the value of k, if 100 sin(2 tan^{-1} (0.75)) = k **[Hint:** $\sin 2\theta = 2\sin\theta\cos\theta$]

37. Prove that:

(a) $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\dfrac{x}{2},\ x\in\left(0,\ \dfrac{\pi}{4}\right)$

(b) $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)=\dfrac{x}{4}-\dfrac{1}{2}\cos^{-1}x$

(c) $\tan^{-1}\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$

(d) $\sin^{-1}\left(2\tan^{-1}\left(\dfrac{2}{3}\right)\right)=\dfrac{12}{13}$

38. (a) Prove that $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]=\sqrt{\dfrac{x^2+1}{x^2+2}}$

(b) Prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)=\dfrac{2b}{a}$

(c) Prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right)=\dfrac{2\sqrt{a^2+b^2}}{b}$.

(d) Prove that : $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x$

(e) Prove that : $\tan^{-1}\left(\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\dfrac{\pi}{4}+\dfrac{x}{2},\ x\in\left(0,\ \dfrac{\pi}{2}\right)$

(f) Prove that : $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\dfrac{x}{2},\ x\in\left(0,\ \dfrac{\pi}{4}\right)$

39. Solve for x :

(a) $\sin^{-1}(6x)+\sin^{-1}(6\sqrt{3}x)=\dfrac{\pi}{2}$

(b) Solve for x : $\sin^{-1}(6x)+\sin^{-1}(6\sqrt{3}x)=\dfrac{-\pi}{2}$

(c) $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\dfrac{5\pi^2}{8}$.

40. Solve for x : $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\dfrac{3}{4}\right),\ x>0$

FIVE MARKS QUESTIONS

Illustration: (For Solving Q.41)

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution: Let, $\cos^{-1} x = A$, $\cos^{-1} y = B$, $\cos^{-1} z = C$

so, $A + B + C = \pi \quad \Rightarrow \quad A + B = \pi - C$

Thus, $\cos(A + B) = \cos(\pi - C)$

$\Rightarrow \quad \cos A \cos B - \sin A \sin B = -\cos C$

$\Rightarrow \quad \cos A \cos B - \sqrt{1-\cos^2 A}\sqrt{1-\cos^2 B} = -\cos C$

$\Rightarrow \quad xy - \sqrt{1-x^2}\ \sqrt{1-y^2} = -z$

$\Rightarrow \quad (xy + z) = \sqrt{1-x^2}\ \sqrt{1-y^2}$

On squaring both the sides, we get

$(xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow \quad \cancel{x^2y^2} + z^2 + 2xyz = 1 - x^2 - y^2 + \cancel{x^2y^2}$

$\therefore \quad \boxed{x^2 + y^2 + z^2 + 2xyz = 1}$

41. Prove the following:

(a) If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, then prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$

(b) If $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$, then prove that $9x^2 + 4y^2 - 12xy\cos\theta = 36\sin^2\theta$.

42. Prove the following:

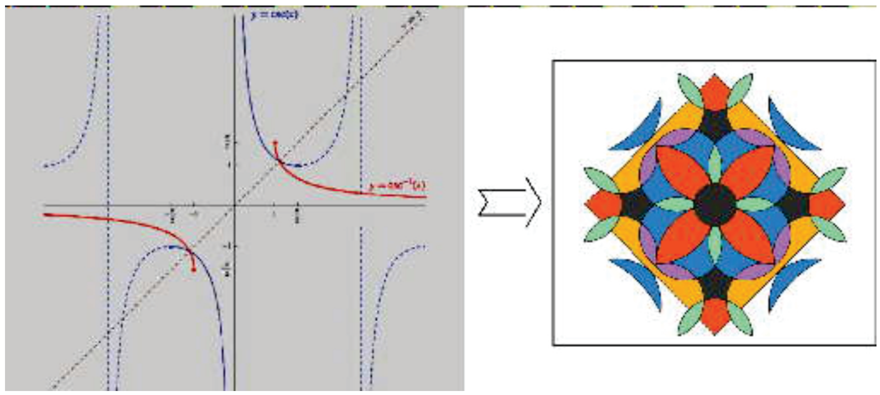
(a) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then prove that $x + y + z = xyz$

(a) If $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \pi$, then prove that $xy + yz + zx = 1$

CASE STUDIES

43. On National Mathematics Day, December 22, 2020, Mathematics Teachers of DOE organized Mathematical Rangoli Competition for the students of all DOE schools to celebrate and remembering the contribution of Srinivasa Ramanujan to the field of mathematics. The legendary Indian mathematician who was born on this date in 1887.

Team A of class XI students made a beautiful Rangoli on Trigonometric Identities as shown in the figure Above, While Team B of class XII students make the Rangoli on the graph of Trigonometric and Inverse Trigonometric Functions. As shown in the following figure.



On the basis of above information, Teacher asked few questions from Team B. Now you try to answer. Those questions which are as follows:

(i) What is the Principal Branch of $\sin^{-1}x$?

(a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) $[0, \pi]$

(ii) What is the Principal Branch of $\cos^{-1}x$?

(a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) $[0, \pi]$

(iii) What is the one Branch of $\operatorname{cosec}^{-1}x$ other than Principal Branch?

(a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ (b) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$

(c) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$ (d) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(iv) If the principal branch $\sec^{-1} x$ is $[0, \pi] - \{k\pi\}$, Then is the value of k?

(a) 0 (b) 1

(c) 0.5 (d) 0.25

(v) What is the Principal Branch of $\tan^{-1}x$?

(a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) $[0, \pi]$

SELF ASSESSMENT-1

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $\cos\left(\cos^{-1}\frac{2}{3}+\sin^{-1}x\right)=0,$ then (3x – 1) = 0

   (a) 0 (b) 1

   (c) –1 (d) 2

2. Domain of the function $\cos^{-1}\left(\frac{x}{2}-1\right)$ is

   (a) [0, 2] (b) [–1, 1]

   (c) [0, 1] (d) [0, 4]

3. If $\cos^{-1}a + \cos^{-1}b = 2\pi$ and $\sin^{-1}c + \sin^{-1}d = \pi$, then $a^2 + b^2 + c^2 + d^2 =$

   (a) 0 (b) 1

   (c) 2 (d) 4

4. The principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

   (a) 0 (b) π

   (c) 2π (d) $\frac{4\pi}{3}$

5. If $\cos^{-1}\left(\frac{1}{x}\right) = \theta$, then $\tan\theta$ =

   (a) x (b) $x^2 + 1$

   (c) $\sqrt{x^2+1}$ (d) $\sqrt{x^2-1}$

SELF ASSESSMENT-2

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2},$ then $(x^3 + y^3 + z^3 - 3xyz) =$

   (a) 0 (b) 1

   (c) –1 (d) 2

2. Principal Range of the function $\sin^{-1}x$ is

   (a) $[0, \pi]$ (b) $(0, \pi)$

   (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

3. If $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)=x,$ then x =

(a) 0 (b) π

(c) $\frac{5\pi}{3}$ (d) $\frac{10\pi}{3}$

4. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2},$ then x =

(a) 0 (b) 1

(c) 2 (d) 3

5. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$ is

(a) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ (B) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$

(C) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (D) $\left[\frac{\pi}{4},\frac{\pi}{2}\right)$

ANSWER

One Mark Questions

1. $\frac{5\pi}{6}$ 2. $\frac{2\pi}{5}$ 3. $\frac{2\pi}{3}$

4. $x=\frac{1}{3}$ 5. $x=\frac{3}{5}$ 6. $\cot^{-1}(-x)=\pi-\cot^{-1}(x)$

7. [0, 1] 8. [1, 2] 9. $\frac{7}{24}$

10. n = 4 (Minimum) 11. x = 1

12. (a) $\frac{\pi}{3}$ (b) π (c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{5}$ (e) $\frac{-\pi}{12}$

13. $\frac{\pi}{5}$ 14. $\sqrt{\frac{2}{3}}$ 15. $\frac{\pi}{2}$

16. $\frac{24}{25}$ 17. $\frac{2\pi}{5}$ 18. (–4)

19. Zero (0) 20. x = 1 21. x = 3

22. Range = $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ 23. A = {0, –3} 24. 2022

25. 2022

Two Marks Questions

26. $A \rightarrow Q$, $B \rightarrow S$, $C \rightarrow S$, $D \rightarrow P$
27. 1
28. $(P^2 + P + 11) = 143$
29. $(P^2 - 2P) = 195$
30. $\frac{1}{\sqrt{5}}$
31. $x = 0$ or -1
32. $\frac{\sqrt{3}}{2}$
33. 1
34. $\frac{\pi}{6}$
35. π

Three Marks Questions

36. 96
39. (a) $x = \frac{1}{12}$ (b) $x = \frac{-1}{12}$ (c) $x = -1$
40. $x = \frac{3}{4}$

CASE STUDIES BASED QUESTION

43. (i) option (a)
43. (ii) option (d)
43. (iii) option (b)
43. (iv) option (c)
43. (v) option (b)

SELF ASSESSMENT-1

1. (b)	2. (d)	3. (d)	4. (b)	5. (d)

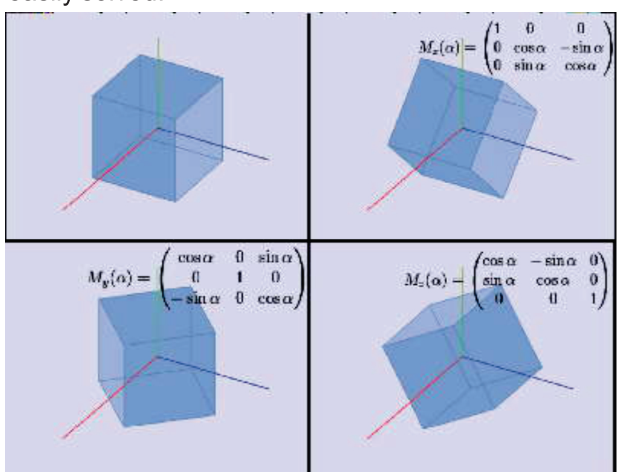
SELF ASSESSMENT-2

1. (a)	2. (c)	3. (a)	4. (d)	5. (c)

CHAPTER-3

MATRICES

Matrices find many applications is scientific field and apply to practical real life problem. Matrices can be solved physical related application and one applied in the study of electrical circuits, quantum mechanics and optics, with the help of matrices, calculation of battery power outputs, resistor conversion of electrical energy into another useful energy, these matrices play a role in calculation, with the help of matrices problem related to Kirchhoff law of voltage and current can be easily solved.



Matrices can play a vital role in the projection of three dimensional images into two dimensional screens, creating the realistic decreeing motion. Now day's matrices are used in the ranking of web pages in the Google search. It can also be used in generalization of analytical motion like experimental and derivatives to their high dimensional.

Matrices are also used in geology for seismic survey and it is also used for plotting graphs. Matrices are also used in robotics and automation in terms of base elements for the robot movements. The movements of the robots are programmed with the calculation of matrices 'row and column' controlling of matrices are done by calculation of matrices.

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23

- Concept, notaion, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices.
- Operation on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Oncommutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2).
- Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

Matrices are defined as a rectangular arrangement of numbers of functions. Since it is a rectangular arrangement, it is 2-dimensional.

A two-dimensional matrix consists of the number of rows (m) and a number of columns (n). Horizontal ones are called Rows and Vertical ones are called columns.

$$A = \begin{pmatrix} M & A & T \\ H & S & I \\ D & O & E \end{pmatrix} \begin{matrix} \longrightarrow \text{Row 1} \\ \longrightarrow \text{Row 2} \\ \longrightarrow \text{Row 3} \end{matrix}$$

Column 1 ↓ Column 2 ↓ Column 3 ↓

ORDER OF MATRIX

The order of matrix is a relationship with the number of elements present in a matrix.

The order of a matrix is denoted by $m \times n$, where m and n are the number of Rows and Columns Respectively and the number of elements in a matrix will be equal to the product of m and n.

TYPES OF MATRICES

Row Matrix

A matrix having only one row is called a row matrix.

Thus $A = [A_{ij}]_{mxn}$ is a row matrix if m = 1. So, a row matirx can be represented as $A = [A_{ij}]_{1xn}$

It is called so because it has only one row and the order of a row matrix will hence be $1 \times n$.

For example,

$A = [1 \quad 2 \quad 3 \quad 4]$ is row matrix of order 1 × 4. Another example of the row matrix is

$B = [0 \quad 9 \quad 4]$ which is of the order 1 × 3.

Column Matrix

A matrix having only one column is called a column matrix. Thus, $A = [A_{ij}]_{mxn}$ is a column matrix if $n = 1$.

Hence, the order is $m \times 1$. An example of a column matrix is:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, B = \begin{pmatrix} M \\ A \\ T \\ H \end{pmatrix}$$

In the above example, A and B are 3 × 1 and 4 × 1 order matrices respectively.

Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

Thus, $A = [A_{ij}]_{mxn}$ is a square matrix if $m = n$; For example is a square matrix of order 3 × 3.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

<u>For Additional Knowledge:</u>

The sum of the diagonal elements in a square matrix A is called the trace of matrix A, and which is sdenoted by tr(A);

$tr(A) = a_{11} + a_{22} + ... + a_{nm}$

Zero or Null Matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generlly denoted by O. Thus, $A = [A_{ij}]_{mxn}$ is a zero-matrix if $a_{ij} = 0$ for all i and j; For example

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here A and B are Null matrix of order 3 × 1 and 2 × 2 respectively.

Diagonal Matrix

If all the non-diagonal elements of a square matrix, are zero, then it is called a diagonal matrix.

Thus, a square matrix $A = [a_{ij}]$ is a diagonal matrix if aij = 0, when $i \neq j$;

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A, B and C are diagonal matrix of order 3 × 3, and D is a diagonal matrix of order 2 × 2.

Diagonal matrix can also be denoted by A = diagonal [2 3 4], B = diag [2 0 4], C = [0 0 4]

<u>Important things to note:</u>

(i) A diagonal matrix is always a square matrix.

(ii) The diagonal elements are characterized by this general form: a_{ij}, where $i = j$. This means that a matrix can have only one diagonal.

Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a scalar matrix.

Thus, a square matrix $A = [a_{ij}]$ is a scalar matrix if

$$A = [a_{ij}] = \begin{cases} 0; & i \neq j \\ k; & i = j \end{cases} \quad \text{Where, } k \text{ is constant.}$$

For example A and B are scalar matrix of order 3 × 3 and 2 × 2 respectively.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$$

Unit Matrix or Identity Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, then it is called a unit matrix.

A unit matrix of order n is denoted by I_n. Thus, a square matrix $A = [a_{ij}]_{m \times n}$ is an identity matrix if

$$A = [a_{ij}] = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}$$

For example I_3 and I_2 are identity matrix of order 3 × 3 and 2 × 2 respectively.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- All identity matrices are scalar matrices
- All scalar matrices are diagonal matrices
- All diagonal matrices are square matrices

Triangular Matrix

A square matrix is said to be a triangular matrix if the elements above or below the principal diagonal are zero. There are two types of Triangular Matrix:

Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an upper triangular matrix, if $a_{ij} = 0$, when $i > j$.

$$A = \begin{pmatrix} D & O & E \\ 0 & D & O \\ 0 & 0 & E \end{pmatrix}$$ is an upper triangular matrix of order 3 × 3.

Lower Triangular Matrix

A square matrix is called a lower triangular matrix, if $a_{ij} = 0$, when $i > j$.

$$A = \begin{pmatrix} D & 0 & 0 \\ O & D & 0 \\ E & O & E \end{pmatrix}$$

is a lower triangular matrix of order 3 × 3.

Transpose of a Matrix

Let A be any matrix, then on interchanging rows and columns of A. The new matrix so obtained is transpose of A donated A^T or A'.

[order of $A = m \times n$, then order of $A^T = n \times m$]

Properties of transpose matrices A and B are:

(a) $(A^T)^T = A$

(b) $(kA)^T = kA^T$ (k = constant)

(c) $(A + B)^T = A^T + B^T$

(d) $(AB)^T = B^T . A^T$

Symmetric Matrix and Skew-Symmetric matrix

- A square matrix $A = [a_{ij}]$ is symmetric if $A^T = A$ i.e. a_{ij} $\forall$ i and j
- A square matrix $A = [a_{ij}]$ is skew-symmetric if $A^T = -A$ i.e. $a_{ij} = -a_{ij}$ $\forall$ i and j

  (All diagonal elements are zero in skew-symmetric matrix)

Illustration:

A is matrix of order 2022 × 2023 and B is a matrix such that AB^T and $B^T A$ are both defined, then find the order of matrix B.

Solution: Let the order of matrix be R × C, So,

$(A)_{2022\times2023} (B^T)_{C\times R}$ $\Rightarrow$ $C = 2023$ (As AB^T is defined)

$(B^T)_{C\times R} (A)_{2022\times2023}$ $\Rightarrow$ $R = 2022$ (As $B^T A$ is defined)

Thus order of matrix B is (2022 × 2023).

Illustration:

If A is a skew symmetric matrix, then show that A^2 is symmetric.

Solution: As A is skew-symmetric, $A^T = -A$

$(A^2)^T = (A.A)^T = A^T.A^T = (-A)(-A) = A^2$

As $(A^2)^T = A^2$

$\Rightarrow$ Thus, A^2 is symmetric.

Illustration:

If $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + X = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, where $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then find the value of $a + c - b - d$.

Solution: As, $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$,

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3-1 & 4+1 \\ 5-2 & 6-3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}$$

On compairing the corresponding elements, we get,

$a = 2$, $b = 5$, $c = 3$, $d = 3$

Thus, $a + c - b - d = 5 - 5 - 3 = -3$

Illustration:

If A is a diagonal matrix of order 3×3 such that $A^2 = A$, then find number of possible matrices A.

Solution: As, A is a diagonal matrix of order 3×3

Let, $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$$\Rightarrow \quad A^2 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

As $A^2 = A \quad \Rightarrow \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$

So, $a = 0$ or -1, similarly b and c can take 2 values (0 and -1).

Thus, total number of possible matrices are $2 \times 2 \times 2 = 8$.

ONE MARK QUESTIONS

1. Write the number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3.

2. Find the Order of the following Matrices. Also find the total Number of elements in each matrix.

   (a) $A = \begin{pmatrix} M & 0 & 0 \\ A & T & 0 \\ H & S & L \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ (b) $C = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

   (c) $D = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 9 \\ 5 & 3 & 0 \\ 6 & 1 & 3 \end{pmatrix}$ (d) $E = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

3. Give an example of a 2 × 2 Non-zero matrices A, B and C such that

   (a) $AB = O$ but $BA \neq 0$ (b) $AB = O$ and $BA = O$ (c) $AB = AC$ but $B \neq C$

4. Give an example of a 3 × 3 matrix which is

   (a) Upper Triangular as well as Lower Triangular Matrix

   (b) Symmetric Matrix

   (c) Skew-Symmetric Matrix

   (d) Neither Symmetric non Skew-Symmetric Matrix

   (e) Symmetric as well as skew-symmetric

5. (a) How many Matrices of order 2 × 2 are possible with entry 0 or 1. How many of these are diagonal matrices. List them.

   (b) How many Matrices of order 3 × 3 are possible with 0 or –1. How many of these are Diagonal matrices?

   (c) If there are five one's i.e. 1, 1, 1, 1, 1 and four zeroes i.e. 0, 0, 0, 0. Thus how many symmetric matrices of order 3 × 3 are possible with these 9 entries?

6. Find 'x', if $A = \begin{pmatrix} 1 & x^2-2 & 3 \\ 7 & 5 & 7 \\ 3 & 7 & -5 \end{pmatrix}$ is symmetric Matrix.

7. Find x, if $A = \begin{pmatrix} 0 & x^2+6 & 1 \\ -5x & x^2-9 & 7 \\ -1 & -7 & 0 \end{pmatrix}$ is skew-symmetric Matrix.

8. Find $(x + y)$, if $A = \begin{pmatrix} 2y-7 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ is scalar Matrix.

9. Find the value of xy, if $A = \begin{pmatrix} 2 & 0 & y-x \\ x+y-2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix.

10. If $(x \quad 1)\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} = O$, then find x.

11. If A is matrix of order $m \times n$ and B is a matrix such that AB' and $B'A$ are both defined, then find the order of matrix B.

12. If A is a skew-symmetric matrix, then $(A^2)^T = kA^2$, find the value of k.

13. If A is a symmetric matrix, then $(A^3)^T = kA^3$, find the value of k.

14. If A is a square matrix such that $A^2 = I$, then find the value of $(A - 1)^3 + (A + I)^3 - 7A$.

15. If $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ find 'k' for $A^2 = kA - 2I$

16. If $A = \begin{pmatrix} 2 & 3 & 5 \\ x-2 & 1 & 6 \\ 6-y & z & 4 \end{pmatrix}$ is an Upper-Triangular matrix, then find the value of $(x + y - z)$.

17. If all entries of a square matrix of order 2 are either 3, –3 or 0, then how many Non-zero matrices are possible?

18. If all the entries of a 3 × 3 Matrix A are either 2 or 6, then how many DIAGONAL matrices are possible?

19. If all the entries of a 3 × 3 Matrix B are either 0 or 1, then how many SCALAR matrices are possible?

20. If all the entries of a 3 × 3 Matrix C are either 0 or 1, then how many IDENTITY matrices are possible?

21. A matrix 'X' has 'p' number of elements, where p is a prime number, then how many orders X can have?

22. Let A and B are two matrices, such that the order of A is 3 × 4, if $A'B$ and BA' are both defined then find the order of B'.

23. If A = diag (3 –5 7), B = diag (–1 2 4) then find $(A + 2B)$.

24. Find the value $(x + y)$ from the following matrix equation:

$$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$$

25. Find Matrix A where, $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

TWO MARKS QUESTIONS

26. If A is a square matrix, then show that
    (a) $(A + A^T)$ is symmetric matrix.
    (b) $(A - A^T)$ is symmetric matrix.
    (c) (AA^T) is symmetric matrix.
27. Show that every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.
28. If A and B are two symmetric matrices of same order, then show that
    (i) $(AB - BA)$ is skew-symmetric Matrix.
    (ii) $(AB + BA)$ is symmetric Matrix.
29. (a) If $A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. Verify that $(A + B)C = AC + BC$.

    (b) If $A + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A - 2B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ then show that $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$
30. If $A = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, show that $AB \neq BA$
31. Find a matrix X, for which $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} \times = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$
32. If A and B are symmetric matrices, show that AB is symmetric, if $AB = BA$.
33. Match the following:

    Possible Number of Matrices (A_n) of order 3 × 3 with entry 0 or 1 which are

	Condition		No. of matrices
(1)	A_n is diagonal Matrix	P	2^0
(2)	A_n is upper triangular Matrix	Q	2^1
(3)	A_n is identity Matrix	R	2^3
(4)	A_n is scalar Matrix	S	2^6

34. If $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ then prove that $A^3 = \begin{pmatrix} \cos 3x & -\sin 3x \\ \sin 3x & \cos 3x \end{pmatrix}$.

35. Express the following Matrices as a sum of a symmetric and skey-symmetric matrix. (Note: Part (b) and (c) can be asked for one marker, SO THINK ABOUT THIS!)

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 7 \\ 5 & 7 & -5 \end{pmatrix}$ (c) $A = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$

36. Show that the Matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A + 1 = 0$.

37. Find the values of x and y, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies the equation $A^2 + xA + yI = 0$.

38. Find $f(A)$, if $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ such that $f(x) = x^2 - 4x + 7$

39. Find A^2 if $A = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$.

40. Find $2A^2$ when $x = \frac{\pi}{3}$ where $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$.

THREE MARKS QUESTIONS

41. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{pmatrix}$ and $Q = [q_{ij}]$ be two 3 × 3 matrices such that $Q = P^5 + I_3$, then Prove that $\left(\frac{q_{21} + q_{31}}{q_{32}}\right) = 10$.

42. Construct a 3 × 3 matrix $A = [a_{ij}]$ such that

(a) $a_{ij} = \begin{cases} i + j; & i > j \\ \frac{i}{j}; & i = j \\ i - j; & i < j \end{cases}$ (b) $a_{ij} = \begin{cases} 2^i; & i > j \\ i.j; & i = j \\ 3^j; & i < j \end{cases}$

(c) $a_{ij} = \begin{cases} i^2 + j^2; & i \neq j \\ 0; & i = j \end{cases}$ (d) $a_{ij} = \frac{|2i - 3j|}{5}$

(e) $a_{ij} = \left[\frac{i}{j}\right]$, where [.] represents Greatest Interger Function.

43. ﬁ $A = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$, then prove that $A^2 = \begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{pmatrix}$, where $i = \sqrt{-1}$

44. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, evaluate $A^3 - 4A^2 + A$.

45. If $f(x) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then prove that $f(x).f(y) = f(x + y)$

46. If $f(x) = \dfrac{1}{\sqrt{1-x^2}}\begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix}$, Prove that $f(x).f(y) = f\left(\dfrac{x+y}{1+xy}\right)$. Hence show that $f(x).f(-x) = 1$, where $|x| < 1$.

FIVE MARKS QUESTIONS

47. Find x, y and z if $A^T = A^{-1}$ and $A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$. Also find how many triplets of (x, y, z) are possible. (NOTE: $A.A^{-1} = A^{-1}A = I$)

48. If A is a symmetric Matrix and B is skew-symmetric Matrix such that $A + B = \begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}$ then show that $AB = \begin{pmatrix} 4 & -2 \\ -1 & -4 \end{pmatrix}$.

49. If $A = \begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}$ and $A^{50} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then show that $(a + b + c + d + 398) = 0$.

CASE STUDIES

41. Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The Quantity of sale (in Kg) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

$$A(\text{September sales}) = \begin{array}{c} \begin{matrix} \text{BASMATI} & \text{PERMAL} & \text{NAURA} \end{matrix} \\ \begin{pmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{pmatrix} \end{array} \begin{matrix} \text{RAMAKRISHAN} \\ \text{GURCHARAN SINGH} \end{matrix}$$

$$B(\text{October sales}) = \begin{array}{c} \begin{matrix} \text{BASMATI} & \text{PERMAL} & \text{NAURA} \end{matrix} \\ \begin{pmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix} \end{array} \begin{matrix} \text{RAMAKRISHAN} \\ \text{GURCHARAN SINGH} \end{matrix}$$

Based on the above information answer the following:

(i) Find the combined sales in September and October for each farmer in each variety.

(a) Total sales $= \begin{pmatrix} 6000 & 3000 & 9000 \\ 7000 & 13000 & 2000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(b) Total sales $= \begin{pmatrix} 6000 & 12000 & 9000 \\ 7000 & 13000 & 2000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(c) Total sales $= \begin{pmatrix} 6000 & 12000 & 9000 \\ 25000 & 13000 & 2000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(d) Total sales $= \begin{pmatrix} 6000 & 12000 & 9000 \\ 25000 & 13000 & 11000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(ii) Find the decrease in sales from September to October.

(a) Net Decrease in sales $= \begin{pmatrix} 4000 & 8000 & 9000 \\ 7000 & 13000 & 2000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(b) Net Decrease in sales $= \begin{pmatrix} 4000 & 8000 & 3000 \\ 15000 & 13000 & 2000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(c) Net Decrease in sales $= \begin{pmatrix} 4000 & 8000 & 3000 \\ 15000 & 7000 & 9000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

(d) Net Decrease in sales $= \begin{pmatrix} 4000 & 8000 & 3000 \\ 15000 & 13000 & 9000 \end{pmatrix}$ (columns: BASMATI, PERMAL, NAURA; rows: RAMAKRISHAN, GURCHARAN SINGH)

If Ramakrishan sell the variety of rice (per kg) i.e. Basmati, Permal and Naura at ₹ 30, ₹ 20 and ₹ 10 respectively. While Gurcharan Singh Sell the variety of rice (per kg) i.e. Basmati, Permal and Naura at ₹ 40, ₹ 30 and ₹ 20 respectively.

(iii) Find the total selling price received by Ramakrishan in the month of September.

(a) ₹ 80,000 (b) ₹ 90,000 (c) ₹ 1,00,000 (d) ₹ 1,10,000

(iv) Find the total selling price received by Gurcharan Singh in the month of September.

(a) ₹ 1,10,000 (b) ₹ 2,10,000 (c) ₹ 3,00,000 (d) ₹ 3,10,000

(v) Find the total selling price received by Ramakrishan in the month of September and October.

(a) ₹ 4,00,000 (b) ₹ 5,00,000 (c) ₹ 5,10,000 (d) ₹ 6,10,000

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If A is a symmetric matrix then which of the following is not Symmetric matrix,

   (a) $A + A^T$ (b) $A.A^T$

   (c) $A - A^T$ (d) A^T

2. Suppose P, Q and R are different matrices of order 3×5, $a \times b$ and $c \times d$ respectively, then value of $ac + bd$ is, if matrix $P + Q - R$ is defined

   (a) 9 (b) 14

   (c) 24 (d) 34

3. If A and B are two square matrices of same order such that, $AB = A$ and $BA = B$, then $(A + B)(A - B) =$

   (a) O (b) A

   (c) $A^2 - B^2$ (d) B

4. If $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$, then $2x + y - z =$

   (a) 1 (b) 3

   (c) 5 (d) 7

5. If a matrix has 2022 elements, how many orders it can have?

   (a) 6 (b) 2

   (c) 4 (d) 8

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If matrix $A = [a_{ij}]_{2\times 2}$ where

   $a_{ij} = \begin{cases} 1, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j \end{cases}$, then $A^{2021} =$

   (a) O (b) A

   (c) $-A$ (d) I

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then $A^4 =$

   (a) A (b) $3A$

   (c) $9A$ (d) $27A$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 + pA + qI = 0$, then pq =

(a) 0 (b) 1

(c) –1 (d) 2

4. If $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$, then $a + b + c + d$ =

(a) 0 (b) 4

(c) 6 (d) 10

5. If A is a square Matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(a) $2A + I$ (b) $A + 2I$

(c) I (d) $A + I$

ANSWER

One Mark Questions

1. 81

2. (a) Order = 3 × 3; 9 (b) Order = 2 × 2; 4 (c) Order = 3 × 1; 3 (d) Order = 4 × 3; 12 (e) Order = 2 × 1; 2

3. Open ended Question (So Any Suitable Answer)

4. Open ended Question (So Any Suitable Answer)

5. (a) 16; 4 (b) 512; 8 (c) 12

6. $x = \pm 3$

7. $x = 3$

8. $x + y = 17$

9. $xy = 1$

10. 2

11. $m \times n$

12. $k = 1$

13. $k = 1$

14. A

15. $k = 1$

16. $x + y - z = 8$

17. 80

18. Zero

19. 2

20. 1

21. Two

22. (4 × 3)

23. diag (1 –1 15)

24. $x + y = 11$

25. $\begin{pmatrix} 8 & -3 \\ 9 & -4 \end{pmatrix}$

Two Marks Questions

31. $X = \begin{pmatrix} -3 & -14 \\ 4 & 17 \end{pmatrix}$

33. $(1) \to R$ $(2) \to S$ $(3) \to P$ $(4) \to Q$

35. (a) $\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 5 & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & -5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{5}{2} \\ 0 & 0 & \frac{11}{2} \\ \frac{-5}{2} & \frac{-11}{2} & 0 \end{pmatrix}$

35. (b) $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 7 \\ 5 & 7 & -5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 35. (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$

37. $x = -2$, $y = 0$ 38. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 39. $\begin{pmatrix} 29 & 24 \\ 6 & 5 \end{pmatrix}$

40. $\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

Three Marks Questions

42. (a) $\begin{pmatrix} 1 & -1 & -2 \\ 3 & 1 & -1 \\ 4 & 5 & 1 \end{pmatrix}$ 42. (b) $\begin{pmatrix} 1 & 9 & 27 \\ 4 & 4 & 27 \\ 8 & 8 & 9 \end{pmatrix}$ 42. (c) $\begin{pmatrix} 0 & 5 & 10 \\ 5 & 0 & 13 \\ 10 & 13 & 0 \end{pmatrix}$

40. (d) $\begin{pmatrix} \frac{1}{5} & \frac{4}{5} & \frac{7}{5} \\ \frac{1}{5} & \frac{2}{5} & 1 \\ \frac{3}{5} & 0 & \frac{3}{5} \end{pmatrix}$ 42. (e) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ 44. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

47. $x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}};$

CASE STUDIES QUESTION

50. (i) option (d) 50. (ii) option (c) 50. (iii) option (c)

50. (iv) option (d) 50. (v) option (c)

SELF ASSESSMENT-1

1. (c) 2. (d) 3. (a) 4. (c) 5. (d)

SELF ASSESSMENT-2

1. (b) 2. (d) 3. (a) 4. (d) 5. (c)

CHAPTER-4

DETERMINANTS

ne of the important aplications of inverse of a non-singular uare matrix is in cryptography.

yptography is an art of communication between two eople by keeping the information not known to others. It based upon two factors, namely encryption and ecryption.

ncryption means the process of transformation of an ormation (plain form) into an unreadable form (coded rm). On the other hand, *Decryption* means the ansformation of the coded message back into original rm. Encryption and decryption require a secret technique hich is known only to the sender and the receiver.

This secret is called a *key*. One way of generating a key is by using a non-singular matrix to encrypt a message by the sender. The receiver decodes (decrypts) the message to retrieve the original mesage by using the inverse of the matrix. The matrix used for encryption is called encryption matrix (encoding matrix) and that used for decoding is called decryption matrix (decoding matrix).

TOPIC TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23

- Determinant of a square matrix (up to 3 × 3 matrice), minors, co-factors and applications of determinants in finding the area of a triangle.
- Adjoint and inverse of a square matrix.
- Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

A determinant of order 2 is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where a, b, c, d are complex numbers (As Complex Number Include Real Number). It denotes the complex number $ad - bc$.

Even though the value of determinnants Represented by Modulus symbol but the value of a determinant may be positive, negative or zero.

In other words,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \text{ (Product of diagonal elements – Product of non-diagonal elements)}$$

- Determinant of order 1 is sthe number itself.
- We can expand the determinants along any Row or Column, but for easier calculations we shall expand the determinant along that row or column which contains maximum number of zeroes.

MINORS AND COFATORS

Minor of an Element

If we take an element of the determinant and delete/remove the row and column containing that element, the determinant of the elements left is called the minor of that element. It is denoted by M_{ij}. For example,

Let us consider a Determinant $|A|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix} \text{ (Minor of } a_{11} = M_{11}\text{)}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix} \text{ (Minor of } a_{11} = M_{11}\text{)}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} \Rightarrow M_{11} = \begin{vmatrix} e & f \\ q & r \end{vmatrix} \text{ (Minor of } a_{11} = M_{11}\text{)}$$

Hence a determinant of order two will have "4 minors" and a determinant of order three will have "9 minors".

Minor of an Element:

Cofactor of the element a_{ij} is $c_{ij} = (-1)^{i+j} M_{ij}$; where i and j denotes the row and column in which the particular element lies. (Means Magnitude of Minor and Cofactor of aij are equal).

- **Property:** If we multiply the elements of any row/column with their respective Cofactors of the same row/column, then we get the value of the determinant.

  For example,

  $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

  $|A| = a_{11}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

- **Property:** If we multiply the elements of any row/column with their respective Cofactors of the other row/column, then we get zero as a result.

  For example,

  $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$

Note that the value of a determinant of order three in term sof 'Minor' and 'Cofactor' can be written as:

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ OR $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Clearly, we see that, if we apply the appropriate sign to the minor of an element, we have its Cofactor. The signs form a check-board pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered, if the row and solumns are inter changed.

$$|A| = |A^T|$$

$$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

- If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only . e.g.

$$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = \begin{vmatrix} b & y & q \\ a & x & p \\ c & z & r \end{vmatrix}$$

- If all the elements of a row (or column) are zero, then the determinant is zero.

$$\begin{vmatrix} a & 0 & x \\ b & 0 & y \\ c & 0 & z \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ x & y & z \end{vmatrix} = 0$$

- If the all elements of a row (or column) are proportional (identical) to the elements sof some other row (or column), then the determinant is zero.

$$\begin{vmatrix} a & ka & x \\ b & kb & y \\ c & kc & z \end{vmatrix} = \begin{vmatrix} mp & mq & mr \\ p & q & r \\ x & y & z \end{vmatrix} = 0$$

- If all the elements of a determinant above or below the main diagonal consist of zeros (Triangular Matrix), then the determinant is equal to the product of diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{vmatrix} = \begin{vmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

- If all the elements of one row/column of a determinant are multiplied by "k" (A scalar), the value of the new determinant is k times the original determinant.

$$\begin{vmatrix} ka & p & x \\ kb & q & y \\ kc & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\begin{vmatrix} ka & kp & x \\ kb & kq & y \\ kc & kr & z \end{vmatrix} = k^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\begin{vmatrix} ka & kp & kx \\ kb & kq & ky \\ kc & kr & kz \end{vmatrix} = k^3 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$|kA| = k^n |A|$, where n is the order of determinant.

AREA OF A TRIANGLE

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ (sq. units)}$$

ADJOINT OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$ be a square matrix and C_{ij} be cofactor of a_{ij} in $|A|$.

Then, (adj A) = $[C_{ij}]$ $\Rightarrow$ adj $A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

- A. (adj A) = (adj A).A = $|A|$
- (adj AB) = (adj B).(adj A)
- |adj A| = $|A|^{n-1}$, where n is the order of a Matrix A

SINGULAR MATRIX

A Matrix A is singular if $|A| = 0$ and it is non-singular if $|A| \neq 0$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5 \neq 0. \text{ So } A \text{ is Non-singular Matrix.}$$

$$|A| = \begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix} = 8 - 8 = 0. \text{ So } A \text{ is singular Matrix.}$$

INVERSE OF A MATRIX

A square matrix A is said to be invertible if there exists a square matrix B of the same order such that $AB = BA = I$ then we write $A^{-1} = B$, (A^{-1} exists only if $|A| \neq 0$)

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

- $(AB)^{-1} = B^{-1}.A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^{T})^{-1} = (A^{-1})^{T}$
- $A.B^{-1} = A^{-1}A = I$
- $|A^{-1}| = \dfrac{1}{|A|}$
- $|A.\text{adj } A| = |A|^{n}$ (Where n is the order of Matrix A)

Illustration:

For what value of k, the matrix $A = \begin{pmatrix} 2 & 10 \\ 5k-2 & 15 \end{pmatrix}$ is singular matrix.

Solution: As, Matrix is singular, so its determinant will be zero.

$|A| = 2(15) - 10(5k - 2) = 30 - 50k + 20$

$|A| = 50 - 50k = 0$

$\Rightarrow\ 50k = 50$

$\therefore\ \boxed{k = 1}$

Illustration:

Without expanding the determinants prove that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution: Let $A = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

We observe here $a_{ij} = -a_{ij}$ (A is skew-symmetric matrix)

$\Rightarrow \; A^T = -A$

$\Rightarrow \; |A^T| = |-A|$

$\Rightarrow \; |A| = (-1)^3 \, |A|$

Property USED: $|A^T| = |A|$, $|kA| = k^n|A|$
Where n is the order of the determinant

$\Rightarrow \; |A| = -|A|$

$\Rightarrow \; 2|A| = 0$

$$\Rightarrow \; |A| = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Illustration:

If A is an invertible matrix of order 2 and $|A| = 4$, then write the value of $|A^{-1}|$.

Solution: As we know that,

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$$

$$\Rightarrow \boxed{|A^{-1}| = \frac{1}{4}}$$

Illustration:

Find the inverse of the matrix $\begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}$ and hence solve the system of equations:

$3x + 4y + 5z = 18$

$5x - 2y + 7z = 20$

$2x - y + 8z = 13$

Solution: Let, $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}$

Cofactors are,

$C_{11} = \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = -7 + 16 = 9$ $\quad C_{21} = -\begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = -38$ $\quad C_{31} = \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} = 37$

$C_{12} = -\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = -(14 - 40) = 26$ $\quad C_{22} = \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -4$ $\quad C_{32} = -\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = -14$

$C_{13} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -4 + 5 = 1$ $\quad C_{23} = -\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 26$ $\quad C_{33} = \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11$

$$\text{Adj } A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}$$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 3(9) + 4(26) + 5(1) = 27 + 104 + 5 = 136$

$$\text{So, } A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{136}\begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}$$

Given system of equation can be written as

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 20 \\ 13 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 4 & 5 \\ 5 & -2 & 7 \\ 2 & -1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 13 \\ 20 \end{pmatrix}$$

$\Rightarrow A.X = B \Rightarrow A^{-1} AX = A^{-1}.B$

$IX = A^{-1}.B \Rightarrow X = A^{-1}.B$

$$X = \frac{1}{136}\begin{pmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{pmatrix}\begin{pmatrix} 18 \\ 13 \\ 20 \end{pmatrix} = \frac{1}{136}\begin{pmatrix} 9\times 18 - 38\times 13 + 37\times 20 \\ 26\times 18 - 4\times 13 - 14\times 20 \\ 1\times 18 + 26\times 13 - 11\times 20 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{136}\begin{pmatrix} 408 \\ 136 \\ 136 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ So, $x = 3$, $y = 1$, $z = 1$

ONE MARK QUESTIONS

1. If A is a matrix of order 3 × 3, then find the value of $|3A|$.
2. What is the sum of the products of elements of any row of a matrix A with the co-factors of corresponding elements.
3. What is the sum of the products of elements of any row of a matrix A with the co-factors of elements of other row.
4. Find the value of k, if the area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units.
5. If A is a square matrix of order 3 × 3 such that $|A| = k$, find the value of $|-A|$.
6. A and B are square matrices of order 3 each, $|A| = 2$ and $|B^T| = 3$. Find $|-4AB|$.
7. If A is an invertible matrix of order 3 and $|\text{Adj } A| = 25$, then write the value of $|5A^{-1}|$.
8. For what value of k, the matrix $\begin{pmatrix} 2 & 10 \\ 3k+2 & -5 \end{pmatrix}$ is singular matrix.
9. Using Determinants, Find the area of triangle with vertices $A(2, 0)$, $B(4, 5)$, $C(6, 3)$.
10. For what value of k, the matrix $\begin{pmatrix} 2 & 5 \\ k & 10 \end{pmatrix}$ has no inverse.
11. If $A = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$, find $A.(\text{Adj } A)$.
12. If $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, find $A.(\text{Adj } A)$.
13. If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then for any natural number n, find the value of $|A^n|$.
14. For what value(s) of a, the points (a, 0), (2, 0) and (4, 0) are collinear?
15. For $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ write A^{-1}.

TWO MARKS QUESTIONS

16. Without expanding the determinants prove that $\begin{vmatrix} 0 & 2023 & -2021 \\ -2023 & 0 & -2022 \\ 2021 & 2022 & 0 \end{vmatrix} = 0$
17. Let A be a 3 × 3 matrix such that $|A| = -2$, then find the value of $|-2A^{-1}| + 2|A|$.

18. If $A = \begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}, B = \begin{pmatrix} yr-zq & cq-br & bz-cy \\ zp-xr & ar-cp & cx-az \\ xq-yp & bp-aq & ay-bx \end{pmatrix}$. Find $|B|$ if $|A| = 4$

19. If $A = \begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}, B = \begin{pmatrix} yr-zq & cq-br & bz-cy \\ zp-xr & ar-cp & cx-az \\ xq-yp & bp-aq & ay-bx \end{pmatrix}$. Find $|A|$ if $|B| = 25$

20. Find the Adjoint of Matrix A,

$$A = \begin{pmatrix} 2\cos\frac{\pi}{3} & -2\sin\frac{\pi}{3} \\ 2\sin\frac{\pi}{3} & -2\cos\frac{\pi}{3} \end{pmatrix}$$

THREE MARKS QUESTIONS

21. If A is a square matrix of order 3, such that $|\text{Adj } A| = 25$, then find the value of

(a) $|A|$ (b) $|-2A^T|$ (c) $|4A^{-1}|$

(d) $|5A|$ (e) $A.\text{Adj } A$ (f) $|A.\text{Adj } A|$

(f) $|A^3|$

22. If A is a square matrix of order 3, such that $|A| = 5$, then find the value of

(a) $|3A|$ (b) $|-2A^T|$ (c) $|4A^{-1}|$

(d) $|\text{Adj } A|$ (e) $A.\text{Adj } A$ (f) $|A.\text{Adj } A|$

(f) $|A^3|$

23. If $A = \begin{pmatrix} 1 & 2020 & 2021 \\ 0 & 1 & 2022 \\ 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 2021 & 1 & 0 \\ 2020 & 2022 & 1 \end{pmatrix}$ then find the value of

(a) $|AB|$ (b) $|(AB)^{-1}|$ (c) $|A^2.B^3|$

(d) $|3(AB)^T|$ (e) $|\text{Adj }(AB)|$

24. Find matrix 'X' such that $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

25. Find matrix 'X' such that

(a) $X\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} X \begin{pmatrix} 7 & -2 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

FOUR/FIVE MARKS QUESTIONS

26. (a) A school wants to award its students for regularity and hardwork with a total cash award of ₹ 6,000. If three times the award money for hardwork added to that given for regularity amounts of ₹ 11,000 represent the above situation algebraically and find the award money for each value, using matrix method.

    (b) A shopkeeper has 3 varieties of pen *A*, *B* and *C*. Rohan purchased 1 pen of each variety for total of ₹ 21. Ayush purchased 4 pens of *A* variety, 3 pens of *B* variety and 2 pen of *C* variety for ₹ 60. While Kamal purchased 6 pens of *A* variety, 2 pens of *B* variety and 3 pen of *C* variety for ₹ 70. Find cost of each variety of pen by Matrix Method.

27. Find A^{-1}, where $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}$. Hence use the result to solve the following system of linear equations:

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + 3y + 2z &= 2 \\ 3x - 3y - 4z &= 11 \end{aligned}$$

28. Find A^{-1}, where $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}$. Hence, solve the system of linear equations:

$$\begin{aligned} x + 2y + 3z &= 8 \\ 2x + 3y - 3z &= -3 \\ -3x + 2y - 4z &= -6 \end{aligned}$$

29. If $A = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ find *AB*. Hence using the product solve the system of eq.

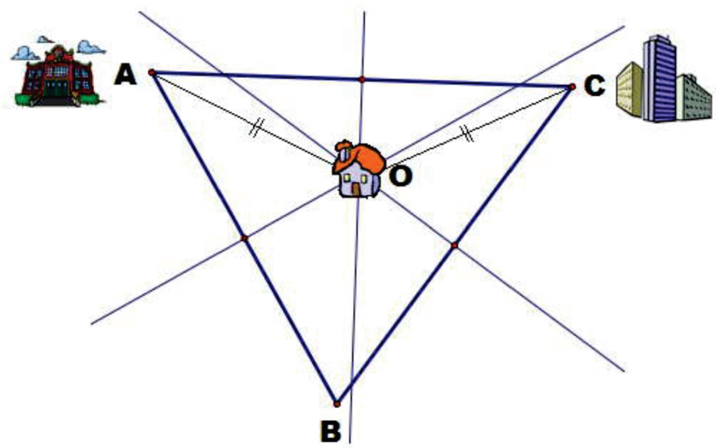
$$\begin{aligned} x - y + z &= 4 \\ x - 2y - 2z &= 9 \\ 2x + y + 3z &= 1 \end{aligned}$$

30. Find the product of matrices *AB*, where $A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{pmatrix}$ and use the result to solve following system of equations:

$$\begin{aligned} x - 2y - 3z &= 1 \\ -2x + 4y + 5z &= -1 \\ -3x + 7y + 9z &= -4 \end{aligned}$$

CASE STUDY BASED QUESTIONS

A. A family wanted to buy a home, but they wanted it to be close both to both the children's school and the parents' workplace. By looking at a map, they cold find a point that is equidistant from both the workplace and the school by finding the *circumcenter* of the triangular region.



If the coordinates are $A(12, 5)$, $B(20, 5)$ and $C(16, 7)$, on the basis of this answer the following: (Figure is for reference only, Not as per scale)

(a) Using the concept of Determinants. Find the equation of AC.

(i) $x - 2y = 0$ (ii) $x + 2y = 22$

(iii) $x + 2y = 30$ (iv) $x - 2y = 2$

(b) Using the concept of Determinants. Find the equation of BC.

(i) $x - 2y = 30$ (ii) $x - 2y = 10$

(iii) $x + 2y = 30$ (iv) $x + 2y = 20$

(c) What will be the area of TRIANGULAR REGION ABC (in sq. Units).

(i) 2 (ii) 4

(iii) 6 (iv) 8

(d) If O (16, 2) is the circum-centre of ΔABC, then find the Area of ΔAOC (in sq. units).

(i) 10 (ii) 6

(iii) 8 (iv) 4

(e) If any point $P(2, k)$ is collinear with point $A(12, 5)$ and $O(16, 2)$, then find the value of $(2k - 15)$.

(i) –10 (ii) 10

(iii) 2 (iv) 0

B. For keeping Fit, X people believes in morning walk, Y people believes in yoga and Z people join Gym. Total no of people are 70. Further 20%, 30% and 40% people are suffering from any disease who believe in morning walk, yoga and GYM respectively. Total no. of such people is 21. If morning walk cost ₹ 0 Yoga cost ₹ 500/month and GYM cost ₹ 400/ month and total expenditure is ₹ 23000.

Solve the above Problem using Matrices and Answer the following:

(a) If we formulate this problem, then which of the equation is NOT possible:

(i) $X + Y + Z = 70$ (ii) $2X + 3Y + 4Z = 21$

(iii) $2X + 3Y + 4Z = 210$ (iv) $5Y + 4Z - 230 = 0$

(b) If matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 4 \end{pmatrix}$, represents the coefficient of x, y and z in above 3 corrected equation, then

(i) $A^{-1} = \frac{-1}{6}\begin{pmatrix} -8 & 1 & 1 \\ -8 & 4 & -2 \\ 10 & -5 & 1 \end{pmatrix}$ (ii) $A^{-1} = \frac{-1}{6}\begin{pmatrix} -8 & 1 & 1 \\ -8 & 4 & -2 \\ 10 & -5 & -1 \end{pmatrix}$

(iii) $A^{-1} = \frac{-1}{6}\begin{pmatrix} -8 & 1 & 1 \\ -8 & 4 & -2 \\ 10 & 5 & -1 \end{pmatrix}$ (iv) $A^{-1} = \frac{-1}{6}\begin{pmatrix} -8 & 1 & 1 \\ -8 & -4 & -2 \\ 10 & -5 & -1 \end{pmatrix}$

(c) On solving above system of equations using matrix method, find the total number of person who prefer Morning Walk.

(i) 10 (ii) 20

(iii) 30 (iv) 40

(d) On solving above system of equations using matrix method, find the total number of person who prefer yoga.

(i) 10 (ii) 20

(iii) 30 (iv) 40

(e) On solving above system of equations using matrix method, find the total number of person who prefer GYM.

(i) 10 (ii) 20

(iii) 30 (iv) 40

C. An amount of ₹ 600 crores is spent by the government in three schemes. Scheme A is for saving girl child from the cruel parents who don't want girl child and get the abortion before her birth.

Scheme B is for saving of newlywed girls from death due to dowry. Scheme C is planning for good health for senior citizen. Now twice the amount spent on Scheme C together with amount spent on Scheme A is ₹ 700 crores. And three times the amount spent on Scheme A together with amount spent on Scheme B and Scheme C is ₹ 1200 crores.

If we assume government invest (In crores) ₹ X, ₹ Y and ₹ Z in scheme A, B and C respectively. Solve the above problem using Matrices and answer the following:

(a) If we formulate this problem, then which of the equation is NOT correct:

(i) $X + Y + Z - 600 = 0$ (ii) $X + 2Z = 700$

(iii) $X + 2Y = 700$ (iv) $3X + Y + Z = 1200 = 0$

(b) If matrix $X = \frac{1}{9}\begin{pmatrix} 5 & -17 \\ -3 & 12 \end{pmatrix}$ represents the coefficient of X, Y and Z in above 3 corrected equation, then

(i) $A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{pmatrix}$ (ii) $A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & 1 & -1 \end{pmatrix}$

(iii) $A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ (iv) $A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 5 & 1 \\ 0 & -2 & -2 \\ 2 & -1 & -1 \end{pmatrix}$

(c) On solving above system of equations using matrix method, find the amount spent (in crores) in Scheme A.

(i) 100 (ii) 200

(iii) 300 (iv) 400

(d) On solving above system of equations using matrix method, find the amount spent (in crores) in Scheme B.

(i) 100 (ii) 200

(iii) 300 (iv) 400

(e) On solving above system of equations using matrix method, find the amount spent (in crores) in Scheme A.

(i) 100 (ii) 200

(iii) 300 (iv) 400

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 5 \end{bmatrix}$, then $|A| =$

(a) 2 (b) 5

(c) 8 (d) 10

2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, then $|A^T| =$

(a) 2 (b) 5

(c) 8 (d) 10

3. If $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then $|A^{-1}| =$

(a) 0 (b) 1

(c) $\cos x . \sin x$ (d) –1

4. If $A = \begin{bmatrix} 6x & 8 \\ 3 & 2 \end{bmatrix}$ is singular matrix, then the value of x is

(a) 2 (b) 3

(c) 5 (d) 7

5. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. The value of k will be

(a) 6 (b) 9

(c) 3 (d) 0

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be

(a) 0 (b) 1

(c) 12 (d) 144

2. If the points (3, –2), (x, 2), (8, 8) are collinear, then x =

(a) 2 (b) 5

(c) 4 (d) 3

3. $\begin{vmatrix} \cos 15^\circ & \sin 75^\circ \\ \sin 15^\circ & \cos 75^\circ \end{vmatrix} =$

(a) 0 (b) 1

(c) –1 (d) 2

4. The minor of 6 in the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is

(a) 9 (b) –6

(c) 6 (d) 10

5. The cofactor of 4 in the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is

(a) 9 (b) –6

(c) 6 (d) 10

ANSWER

One Mark Questions

1. $27|A|$
2. $|A|$
3. 0
4. ±3
5. $-k$
6. –384
7. ±25
8. (–1)
9. 7 sq. units
10. $k = 4$
11. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
12. $\begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$
13. 1
14. a can be any real number
15. $\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$

Two Marks Questions

17. 0
18. 16
19. ±5
20. $\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

Three Marks Questions

21. (a) ±5 (b) ±40 (c) $\frac{\pm 64}{5}$ (d) ±625 (e) $\pm 5I$

22. (a) 135 (b) –40 (c) $\frac{64}{5}$ (d) 25 (e) $5I$

(f) 125 (e) 125

23. (a) 6 (b) $\frac{1}{6}$ (c) 72 (d) 62 (e) 36

24. $X = \frac{1}{9}\begin{pmatrix} 2 & 31 \\ -1 & -11 \end{pmatrix}$

25. (a) $X = \begin{pmatrix} 16 & -25 \\ 1 & -1 \end{pmatrix}$ (b) $X = \begin{pmatrix} 11 & -7 \\ -5 & 4 \end{pmatrix}$ (c) $X = \frac{1}{9}\begin{pmatrix} 5 & -17 \\ -3 & 12 \end{pmatrix}$

Three Marks Questions

26. (a) Award money given for Honesty = ₹ 500, Regularity = ₹ 2000 and Hard work = ₹ 3500

    (b) Cost of pen of Variety A = ₹ 5, Variety B = ₹ 8 and Variety C = ₹ 8

27. $x = 3$, $y = -2$, $z = 1$ 28. $x = 0$, $y = 1$, $z = 2$ 29. $x = 3$, $y = -2$, $z = -1$

30. $x = -4$, $y = -1$, $z = -1$

CASE STUDIES QUESTIONS

A. (a) option (iv) (b) option (iii) (c) option (iv)
   (d) option (i) (e) option (i)

B. (a) option (ii) (b) option (i) (c) option (ii)
   (d) option (iii) (e) option (ii)

C. (a) option (iii) (b) option (ii) (c) option (iii)
   (d) option (i) (e) option (ii)

SELF ASSESSMENT-1

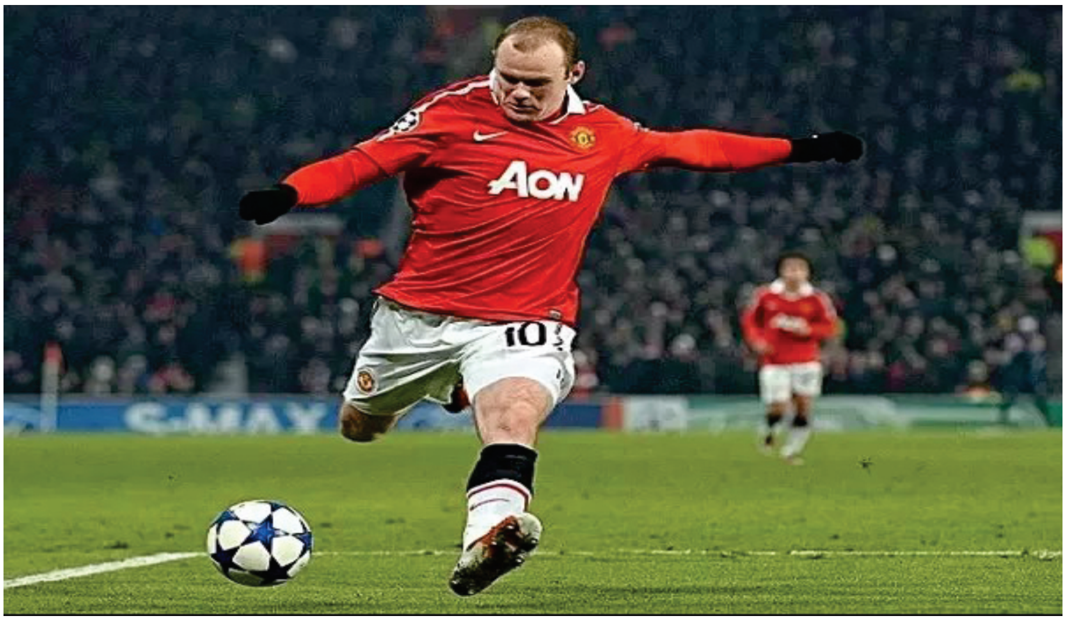
1. (d) 2. (d) 3. (b) 4. (a) 5. (c)

SELF ASSESSMENT-2

1. (d) 2. (b) 3. (a) 4. (b) 5. (c)

CHAPTER 5

CONTINUITY AND DIFFERENTIABILITY



Many real life events, such as trajectory traced by Football where you see player hit the soccer ball, angle and the distance covered animation on the screen is shown to the viewers using technology can be described with the help of mathematical functions. The knowledge of Continuity and differentiation is popularly used in finding speed, directions and other parameters from a given function.

CONTINUITY AND DIFFERENTIABILITY

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

- Continuity and differetiability
- Chair rule
- Derivative of inverse trigonometric functions, like $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$
- Concept of exponential and logarithmic function
- Derivatives of logarithmic and exponential functions.
- Logarithmic differentiation, derivative of functions expressed in parametric forms.
- Second order derivatives.

POINTS TO REMEMBER

- A function $f(x)$ is said to be continuous at $x = c$ iff $\lim_{x\to c} f(x) = f(c)$

  i.e., $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

- $f(x)$ is continuous in (a, b) iff it is continuous at $x = c\ \forall c \in (a,b)$.

- $f(x)$ is continuous in $[a, b]$ iff

  (i) $f(x)$ is continuous in (a, b)

  (ii) $\lim_{x\to a^+} f(x) = f(a)$

  (iii) $\lim_{x\to b^-} f(x) = f(b)$

- Modulus functions is Continuous on R

- Trigonometric functions are continuous in their respective domains.

- Exponential function is continuous on R

- Every polynomial function is continuous on R.

- Greatest integer function is continuous on all non-integral real numbers

- If $f(x)$ and $g(x)$ are two continuous functions at $x = a$ and if $c \in R$ then

  (i) $f(x) \pm g(x)$ are also continuous functions at $x = a$.

  (ii) $g(x) . f(x)$, $f(x) + c$, $cf(x)$, $|f(x)|$ are also continuous at $x = a$.

  (iii) $\frac{f(X)}{g(X)}$ is continuous at $x = a$, provided $g(a) \neq 0$.

- A function $f(x)$ is derivable or differentiable at $x = c$ in its domain iff

  $\lim_{x\to c^-} \frac{f(X)-f(c)}{X-c} = \lim_{x\to c^+} \frac{f(X)-f(c)}{X-c}$, and is finite

  The value of above limit is denoted by $f'(c)$ and is called the derivative of $f(x)$ at $x = c$.

  $\frac{d}{dx}(u \pm v) = \frac{du}{dx} + \frac{dv}{dx}$

- $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$ (Product Rule)

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ (Quotient Rule)

- If y= f(u) and u=g(t) then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u)g'(t)$ (Chain Rule)

- If y =f(u), x=g(u) then,

  $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}$$

Illustration:

Discuss the continuity of the function *f*(*x*) given by

$f(x) + \begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$ at $x = 4$

Solution: We have $f(x) = \begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$

$$\text{LHL} = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (4 - x) = \lim_{x \to 0^-} 4 - (4 - h) = 0$$

$$\text{RHL} = \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (4 + x) = \lim_{x \to 0} 4 + (h + 4) = 8 + 0 = 8$$

Here LHL ≠ RHL

Hence *f*(*x*) is not continuous at *x* = 4

Illustration:

Show that the function *f*(*x*) given by

$f(x) + \begin{cases} \frac{\tan x}{x} + \cos x, & x \neq 4 \\ 2, & x = 4 \end{cases}$ is continuous at $x = 0$

Solution: We have $f(x) = f\begin{cases} \frac{\tan x}{x} + \cos x, & x \neq 4 \\ 2, & x = 4 \end{cases}$

Now *f*(0) = 2

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\tan x}{x} + \cos x\right) = \lim_{h \to 0^-} \frac{\tan(0 - h)}{(o - h)} + \cos(0 - h) - \lim_{h \to 0} \frac{-\tan h}{-h} + \cos h$$

$$= \lim_{h \to 0} \frac{\tan h}{h} + \lim_{h \to 0} \cos h = 1 + \cos(0) = 1 + 1 = 2$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{\tan x}{x} + \cos x\right) = \lim_{h\to 0^+} \frac{\tan(0-h)}{(0-h)} + \cos(0-h)$$

$$= \lim_{h\to 0} \frac{\tan h}{h} + \lim_{h\to 0} \cos h = 1 + \cos(0) = 1 + 1 = 2$$

$$\text{LHL} = \text{RHL} = f(0)$$

Hence $f(x)$ is continuous at $x = 0$

ONE MARK QUESTIONS

1. Let f(x) = sin x cos x. Write down the set of points of discontinuity of f(x).

2. Given f (x) = $\frac{1}{x+2}$, write down the set of points of discontinuity of f (f(x)).

3. Write the set of points of continuity of

   $f(x) = |x-1| + |x+1|$

4. Write the number of points of discontinuity of $f(x) = [x]$ in [3, 7].

5. If $y = e^{\log(x^5)}$, find $\frac{dy}{dx}$.

6. If $f(x) = x^2 g(x) and\ g(1) = 6,\ \ g'(x) = 3, find\ the\ value\ of f'(1)$.

7. If $y = a \sin t$, $x = a \cos t$ then find $\frac{dy}{dx}$

8. Find value of f(0), so that $\frac{-e^x+2^x}{x}$ may be continuous at x=0.

9. Find the values of x for which f(x) = $\frac{x^2+7}{x^3+3x^2-x-3}$ is discontinous.

10. If y = $\tan^{-1}x + \cot^{-1}x + \sec^{-1}x$, $\text{cosec}^{-1}x$ then find dy/dx

11. If y = $\log_e e^{\sin x^2}$, find $\frac{dy}{dx}$

12. y = $\log_a x + \log_x a + \log_x x + \log_a a$, then $\frac{dy}{dx}$ = ?

    (a) $\frac{1}{x}$ + x loga (b) $\frac{1}{x \log a}$ + x loga (c) $\frac{\log a}{x} + \frac{x}{\log a}$ (d) None of these

13. If $y = 5^x \cdot x^5$, then find $\frac{dy}{dx}$

14. What is derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\sin^{-1}(3x - 4x^3)$?

15. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$, then find $(2y - 1)\frac{dy}{dx}$

TWO MARKS QUESTIONS

1. Differentiate $\sin(x^2)$ w. r. t. $e^{\sin x}$

2. $y = x^y$ then find $\frac{dy}{dx}$

3. If $y = x^x + x^3 + 3^x + 3^3$, find $\frac{dy}{dx}$

4. If $y = 2\sin^{-1}(\cos x) + 5\,\mathrm{cosec}^{-1}(\sec x)$. Find $\frac{dy}{dx}$

5. If $y = e^{[\log(x+1) - \log x]}$ find $\frac{dy}{dx}$

6. Differentiate $\sin^{-1}[x\sqrt{x}]$ w. r. t. x.

7 Find the derivative of $|x^2+2|$ w.r.t. x

8. Find the domain of the continuity of $f(x) = \sin^{-1}x - [x]$

9. Find the derivative of $\cos(\sin x^2)$ w.r.t. x at $x = \sqrt{\frac{\pi}{2}}$

10. If $y = e^{3\log x + 2x}$, Prove that $\frac{dy}{dx} = x^2(2x+3)\,e^{2x}$.

11. Differentiate $\sin^2(\theta^2+1)$ w.r.t. θ^2

12. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$

13. If $x^2 + y^2 = 1$ verify that $\frac{dy}{dx}.\frac{dx}{dy} = 1$

14. Find $\frac{dy}{dx}$ when $y = 10^{x10^x}$

15. If $y = x^x$ find $\frac{d^2y}{dx^2}$

16. Find $\frac{dy}{dx}$ if $y = \cos^{-1}(\sin x)$

17. If $f(x) = x + 7$, and $g(x) = x - 7$, $x \in R$, them find $\frac{d}{dx}$ (fog) (x).

18. Differntiate log (7 logx) w.r.t x

19. If $y = f(x^2)$ and $f'(x) = \sin x^2$. Find $\frac{dy}{dx}$

20. Find $\frac{dy}{dx}$ if $y = \sqrt{\sin^{-1}\sqrt{x}}$

THREE MARKS QUESTIONS

1. Examine the continuity of the following functions at the indicated points.

(I) $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(II) $f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0, & x = 1 \end{cases}$ at $x = 1$

(III) $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(IV) $f(x) = \begin{cases} \frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)} & x \neq \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & x = \frac{1}{\sqrt{2}} \end{cases}$ at $x = \frac{1}{\sqrt{2}}$

2. For what values of constant K, the following functions are continuous at the indicated points.

(i) $f(x) = \begin{cases} \frac{\sqrt{1+Kx}-\sqrt{1-Kx}}{x} & x < 0 \\ \frac{2x+1}{x-1} & x > 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)} & x \neq 0 \\ K & x = 0 \end{cases}$ at $x = 0$

(iii) $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & x < 0 \\ K & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & x > 0 \end{cases}$ at $x = 0$

3. For what values a and b

$$f(x) = \begin{cases} \dfrac{x+2}{|x+2|} + a & if\ x < -2 \\ a + b & if\ x = -2 \\ \dfrac{x+2}{|x+2|} + 2b & if x > -2 \end{cases}$$

Is continuous at $x = -2$

4. Find the values of a, b and c for which the function

$$f(x) = \begin{cases} \dfrac{\sin[(a+1)x] + sinx}{x} & x < 0 \\ C & x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$$

Is continuous at $x = 0$

5. $f(x) = \begin{cases} [x] + [-x] & x \neq 0 \\ \lambda & x = 0 \end{cases}$

Find the value of λ, *f* is continuous at $= 0$?

6. Let $f(x) = \begin{cases} \frac{1 - sin^3 x}{3co\ ^2 x} & ; \quad x < \frac{\pi}{2} \\ a & ; \quad x = \frac{\pi}{2} \\ \frac{b(1 - \sin\ \ )}{(\pi - 2x)^2} & ; \quad x > \frac{\pi}{2} \end{cases}$

If $f(x)$ is continuous at $x = \frac{\pi}{2}$, find *a* and *b.*

7. If $f(x) = \begin{cases} x^3 + 3x + a & x \leq 1 \\ bx + 2 & x > 1 \end{cases}$

Is everywhere differentiable, find the value of *a* and *b*.

8. Find the relationship between a and b so that the function defined by

$$f(x) = \begin{Bmatrix} ax + 1 \ , & x \leq 3 \\ bx + 3 \ , & x > 3 \end{Bmatrix} \textit{is continous at } x = 3.$$

9. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ where $x \neq 0$.

10. If $y = x^{x^x}$, then find $\frac{dy}{dx}$.

11. Differentiate $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$ w.r.t. x.

12. If $(x+y)^{m+n} = x^m . y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

13. If $(x-y).e^{\frac{x}{x-y}} = a$, prove that $y\left(\frac{dy}{dx}\right) + x = 2y$

14. If $x = \tan\left(\frac{1}{a}\log y\right)$ then show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

15. If $y = x\log\left(\frac{x}{a+bx}\right)$ prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

16. Differentiate $\sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$ w.r.t x.

17. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$, prove that

$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$, Where $-1 < x < 1$ and $-1 < y < 1$ [HINT: put $x^3 = \sin A$ and $y^3 = \sin B$]

18. If $f(x) = \sqrt{x^2+1}, g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x-3$ find $f'[h'(g'(x))]$.

19. If $x = \sec\theta - \cos\theta$ and $y = sec^n\theta - cos^n\theta$, then prove that $\frac{dy}{dx} = n\sqrt{\frac{y^2+4}{x^2+4}}$

20. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

21. If $x = a\,cos^3\theta, y = a\,sin^3\theta$ then find $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{6}$

22. If $y = \tan^{-1}\left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right]$ where $0 < x < \frac{\pi}{2}$ find $\frac{dy}{dx}$

23. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,then show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

24. If $= \left[x + \sqrt{x^2+1}\right]^m$, show that $(x^2+1)y_2 + xy_1 - m^2y = 0$.

25. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log\ \ )^2}$

26. If $y^{1/m} + y^{-1/m}$ = 2x then prove that $(x^2-1)y_2 + xy_1 = m^2y$.

<u>ANSWERS</u>

ONE MARK QUESTIONS

1. { }
2. $\{-2, -5/2\}$
3. R
4. Points of discontinuity of $f(x)$ are 4,5,6,7

Note- At $x = 3, f(x) = [x]$ is continuous because $\lim_{x \to 3^+} f(x) = 3 = f(3)$

5. $5x^4$
6. 15
7. $-\cot t$
8. -1+log2
9. x=-1,1,-3
10. 0
11. 2x cos (x^2)
12. (d)
13. $5^x(x^5\log 5 + 5x^4)$
14. 2/3
15. (c) $\frac{\cos x}{(2y-1)}$

TWO MARKS QUESTIONS

1. $\frac{2x\cos(x^2)}{\cos x\, e^{\sin x}}$
2. $\frac{y^2}{x[1-y\log x]}$
3. x^x [1 + log x] + $3x^2$ + 3^x $\log_e 3$
4. –7
5. $-\frac{1}{x^2}$
6. $\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$
7. $\frac{2x(x^2+2)}{|x^2+2|}$
8. $(-1,0) \cup (0,1)$
9. 0
11. $\text{Sin}\,(2\theta^2+2),\ \theta \neq 0$
12. 0
14. $10^{x^{10^x}} 10^x \log 10(1+x\log 10)$
15. $x^x[1+\log x]$
16. -1
17. 1
18. $\frac{1}{x\log x}$
19. 2x $\sin x^4$
20. $\frac{1}{4\sqrt{x}\sqrt{1-x}\sqrt{\sin^{-1}\sqrt{x}}}$, where 0 < x < 1

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If $y = \sin^2 x - \cos^2 x$, then $\frac{dy}{dx} =$

   (a) $2 \sin x$ (b) $2 \cos x$

   (c) $2 \sin 2x$ (d) $-2 \sin 2x$

2. The value of '4k' for which the function $f(x)$ is continuous at $x = 3$.

$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & \text{when } x \neq 3 \\ 2k+1, & \text{when } x = 3 \end{cases}$$

   (a) 4 (b) 6

   (c) 11 (d) 22

3. Derivative of $\sin x$ with respect to $\cos x$ is

   (a) $\tan x$ (b) $-\tan x$

   (c) $\cot x$ (d) $-\cot x$

4. If $y=(x+\sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx}+x\frac{dy}{dx}=$

(a) n^2y (b) ny

(c) y (d) $-ny$

5. If $x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$ then $\frac{d^2y}{dx^2}=$

(a) $\frac{\sec^3\theta}{a}$ (b) $\frac{\sec^3\theta}{a\theta}$

(c) $\sec^3\theta$ (d) $\theta\sec^3\theta$

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. A Function defined as

$$f(x)=\begin{cases}|x|+3, & \text{when } x<0\\ 5-|x|, & \text{when } x\geq 0\end{cases}$$

is continuous on

(a) R (b) $R-\{0\}$

(c) $[0,\infty)$ (d) $(-\infty, 0]$

2. The function g(x) = (sin x + cos x) is continuous at

(a) R (b) $R-\{0\}$

(c) $R-\left\{\frac{\pi}{2}\right\}$ (d) $R-\{\pi\}$

3. The value of the derivative of $|x-2|+|x-3|$ at $x=2$ is

(a) 1 (b) 3

(c) 2 (d) 0

4. If $\sin y = x.\cos(a+y)$ then $\frac{dy}{dx}=$

(a) $\frac{\cos^2(a+y)}{\cos a}$ (b) $\frac{\cos^2(a+y)}{\sin a}$

(c) $\frac{\sin^2(a+y)}{\cos a}$ (d) $\frac{\sin^2(a+y)}{\sin a}$

5. If $y=\left(\frac{x^a}{x^b}\right)^{a+b}.\left(\frac{x^b}{x^c}\right)^{b+c}.\left(\frac{x^c}{x^a}\right)^{c+a}$, then $\frac{dy}{dx}=$

(a) 1 (b) abc

(c) $a+b+c$ (d) 0

THREE MARKS QUESTIONS

1. (I) Continuous (II) Discontinuous
   (III) Not Continuous at $x = 0$ (IV) Continuous
2. (I) K = -1 (II) K = $^1/_2$
   (III) K = 8
3. $a = 0,\ b = -1$
4. $a = \frac{-3}{2},\ b = R - \{0\},\ c = \frac{1}{2}$
5. $\lambda = -1$
6. $a = \frac{1}{2},\ b = 4$
7. $a = 3,\ b = 5$
8. 3a – 3b = 2
9. $-\frac{1}{2}$
10. $x^x x^{x^x}\left\{(1 + \log x) \log x + \frac{1}{x}\right\}$
11. $(x \cos x)^x[1 - x\ \tan x + (\log x\ \cos x)] + (x \sin x)^{1/x}\left[\frac{1+x\ cot\,x - log(x\ sin)^x}{x^2}\right]$
16. $\left[\frac{2^{x+1}3^x}{1+(36)^x}\right] \log 6$
18. $\frac{2}{\sqrt{5}}$
20. $\frac{dy}{dx} = \frac{x^x(1+\log x)+yx^{y-1}-y^x \log y}{x^y \log x + xy^{x-1}}$
21. $\frac{32}{27a}$
22. $-\frac{1}{2}$

SELF ASSESSMENT TEST-1

1. (C) 2. (C) 3. (D) 4. (A) 5. (B)

SELF ASSESSMENT TEST-2

1. (B) 2. (A) 3. (C) 4. (A) 5. (D)

CHAPTER 6

APPLICATION OF DERIVATIVES

The sight of soap bubble produced using a bubble wand is very exciting! One application of derivative is finding the rate of increase of size of the bubble (*dv*/*dt*) due to increasing radius, where *V* is the volume of spherical bubble and *r* is the radius. This can be calculated by knowing the rate of increase of radius with time (*dr*/*dt*).

APPLICATION OF DERIVATIVES

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

- Applications of derivatives:
- rate of change of bodies,
- increasing/decreasing functions,
- maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool).
- Simple problems (that illustrate basic principles and understanding of the subject as well as real life situations).

POINTS TO REMEMBER

- **Rate of change:** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left\{\frac{dy}{dx}\right\}_{x=x_1}$ or f' (x_1)represents the rate of change of y w.r.t. x at $x = x_1$.

- **Increasing and Decreasing Function**

  Let f be a real-valued function and let I be any interval in the domain of f. Then f is said to be

  a) Strictly increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

  b) Increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

  c) Strictly decreasing in I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

  d) Decreasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- **Derivative Test:** Let f be a continuous function on [a, b] and differentiable on (a, b). Then

  a) f is strictly increasing on [a, b] if $f'(x) > 0$ for each $x \in (a, b)$.

  b) f is increasing on [a, b] if $f'(x) \geq 0$ for each $x \in (a, b)$.

  c) f is strictly decreasing on [a, b] if $f'(x) < 0$ for each $x \in (a, b)$.

d) f is decreasing on [a, b] if $f'(x) \le 0$ for each $x \in (a, b)$.

e) f is constant function on [a, b] if $f'(x) = 0$ for each $x \in (a, b)$.

- **Maxima and Minima**

a) Let f be a function and c be a point in the domain of f such that either f '(x)=0 or f '(x) does not exist are called critical points.

b) **First Derivative Test:** Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in I. Then

i. $f'(x)$ changes sign from positive to negative as x increases through *c*, then *c* is called the point of the local maxima.

ii. $f'(x)$ changes sign from negative to positive as *x* increases through *c*, then *c* is a point of *local minima*.

iii. $f'(x)$ does not change sign as *x* increases through *c*, then *c* is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

c) **Second Derivative Test :** Let *f* be a function defined on an interval I and let $c \in$ I. Let f be twice differentiable at c. Then

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of *f*.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value $f(c)$ is local minimum value of *f*.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

EXTREME VALUE OF A FUNCTION

Let $y = f(x)$ be a real function defined on an interval I and *C* be any point in I. Then *f* is said to have an extreme value in *I* if $f(c)$ is either maximum or minimum value of *f* in *I*.

Here, $f(c)$ is called the extreme value and *C* is called one of the extreme points.

Illustration:

Let $f(x) = (2x - 1)^2 + 3$.

Then, $f(x) \geq 3$, as $(2x - 1)^2 \geq 0$

For any real number 'x'

$\Rightarrow (2x - 1)^2 + 3 \geq 0 + 3$

Thus, minimum value of $f(x)$ is 3, which occurs at $x = \frac{1}{2}$

Also $f(x)$ has no maximum value as $f(x) \to \infty$ as $|x| \to \infty$

Illustration:

Let $g(x) = -(x - 1)^2 + 10$.

Then, $g(x) = 10 - (x - 1)^2 \leq 10 \quad \forall x \in R$ as $(x - 1)^2$ is

Always greater them or equal to zero.

Thus maximum value of $g(x)$ is 10, which occurs at $x = 1$

Also $g(x)$ has no minimum value of $f(x) \to -\infty$ as $|x| \to \infty$.

Illustration:

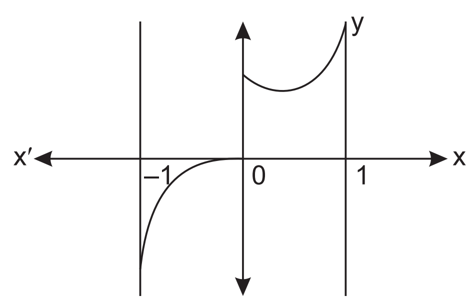
Neither maximum nor minimum value of a function.

Let as consider a function $f(x) = x^3$, $x \in (-1, 1)$

Since this function is an increasing function in (–1, 1), it should have minimum value at a point nearest to –1 and maximum value at a point nearest to 1.

But we can not locate such points (see figure)

So, $f(x) = x^3$, has neither maximum nor-minimum value in (–1, 1).



But, if we extend the domain of f to [–1, 1], then the function $f(x) = x^3$ has maximum value 1 at $x = 1$ and minimum value –1 at $x = -1$

Note: Every continuous function on an closed interval has a maximum and minium value.

ONE MARK QUESTIONS

1 Find the angle θ, where 0< θ<$\frac{\pi}{2}$, which increases twice as fast as its sine.

2. A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with respect to its radius when the radius is 7cm.

3. Write the interval for which the function $f(x) = \cos x, 0 \leq x \leq 2\pi$ is decreasing

4. For what values of x is the rate of increasing of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

5. Write the maximum value of $f(x) = \frac{\log x}{x}$,if it exists.

6. Find the least value of $f(x) = ax + \frac{b}{x}$, where a>0, b>0 and x>0.

7. Find the interval in which the function $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right)$ increases

8. Find the value of a for which the function $f(x) = x^2 - 2ax + 6, x > 0$ is strictly increasing.

9. Find the minimum value of sin x + cos x.

10. Show that f(x) = cos 2x is Decreasing on $\left(0, \frac{\pi}{2}\right)$

11. Find the absolute maximum of $x^{40} - x^{20}$ on the interval (0,1).

12. Find the angle between $y^2 = x$ and $x^2 = y$ at the origin.

13. Find the local minimum value of f'(x) if f(x) = 3 + | x |, x $\in$ R.

14. The distance covered by a particle in t sec.is given by $x = 3 + 8t - 4t^2$. What will be its velocity after 1 second.

15. If the rate of change of volume of a sphere is equal to the rate of change of its radius, then find r.

TWO MARKS QUESTIONS

1. The sum of the two numbers is 8, what will be the maximum value of the sum of their reciprocals.

2. Find the maximum value of $f(x) = 2x^3 - 24x + 107$ in the internal [1, 3]

3. If the rate of change of Area of a circle is equal to the rate of change its diameter. Find the radius of the circle.

4. The sides of on equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when side is 10 cm.

5. If there is an error of a% in measuring the edge of cube, then what is the percentge error in its surface?

6. If an error of k% is made in measuring the radius of a sphere, then what is the percentage error in its volume?

7. If the curves $y = 2e^x$ and $y = ae^{-x}$ interect orthogonally, then find a.

8. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

9. Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$.

10. Find the point on the curve $y = x^2$, where the slope of the tangent is equal to the x co-ordinate of the point.

11. Use differentials to approximate the cube root of 66.

12. Find the maximum and minimum values of the function $f(x) = \sin(\sin x)$

13. Find the local maxima and minima of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$.

14. If $y = a \log x + bx^2 + x$ has its exteme values at $x = -1$ and $x = 2$, then find a and b.

15. If the radius of the circle increases from 5 into 5.1 cm, then find the increase in area.

THREE MARKS QUESTIONS

1. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

2. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $\frac{3}{2}$c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

3. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.

4. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.

5. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10-t)^2$. How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?

6. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.

7. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is $\tan^{-1}(0.5)$. water is poured into it at a constant rate of$5m^3/h$. Find the rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

8. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

9. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of $1.5m^3/min$. find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

10. Let x and y be the sides of two squares such that $y = x - x^2$. Find the rate of change of area of the second square w.r.t. the area of the first square.

11. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

12. If the areas of a circle increases at a uniform rate, then prove that the perimeter various inversely as the radius.

13. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all $x \in R$. Find its value when the rate of increase of f(x) is least.

[Hint: Rate of increase is least when $f'(x)$ is least.]

14. Determine whether the following function is increasing or decreasing in the given interval: $f(x) = \cos\left(2x + \frac{\pi}{4}\right), \frac{3\pi}{8} \leq x \leq \frac{5\pi}{8}$.

15. Determine for which values of x, the function y=$x^4 - \frac{4x^3}{3}$ is increasing and for which it is decreasing.

16. Find the interval of increasing and decreasing of the function $f(x) = \frac{\log x}{x}$

17. Find the interval of increasing and decreasing of the function $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$.

18. Show that $f(x) = x^2 e^{-x}, 0 \leq x \leq 2$ is increasing in the indicated interval.

19. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

20. Find the intervals in which the following functionis decreasing.

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

21. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

22. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$,is strictly increasing the interval $\left(0, \frac{\pi}{4}\right)$.

23. Find the interval in which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is increasing or decreasing.

24. Find the interval in which the function given by

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

(i) strictly increasing

(ii) strictly decreasing

25. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

26. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when x=3?

27. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

28. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.

29. Find the maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

30. Find the absolute maximum value and absolute minimum value of the following question $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$ in [-2, 2.5]

31. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval [0, 1]

32. Find the absolute maximum and absolute minimum value of $f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

33. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

FIVE MARKS QUESTIONS

1. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

2. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

3. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

4. The sum of the surface areas of cuboids with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x = 3 radius of the sphere. Also find the minimum value of the sum of their volumes.

5. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

6. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

7. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

8. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle $\alpha\ is\ \frac{4}{27}\pi h^3 tan^2\alpha.$ Also show that height of the cylinder is $\frac{h}{3}$

9. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2,1).

10. Find the shortest distance between the line $y - x = 1$ and the curve$x = y^2$.

11. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

13. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$

   (a) 72 units/sec (b) –72 units/sec

   (c) 54 units/sec (d) –54 units/sec

2. The function $f(x) = \tan x - 4x$, on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ is

   (a) strictly decreasing (b) strictly increasing

   (c) neither increasing or deceasing

   (d) Non of these

3. The curve $y = xe^x$ has minimum value equal to

(a) 1 (b) 0

(c) –e (d) $-\frac{1}{e}$

4. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate (in cm^2/sec) at which the area increases, when side is 10 cm is

(a) 10 (b) 5

(c) $10\sqrt{3}$ (d) $5\sqrt{3}$

5. If $ab = 2a + 3b$, $a > 0$, $b > 0$ then the minimum value of ab is

(a) 6 (b) 12

(c) 24 (d) 48

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$ where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a =

(a) 0 (b) 1

(c) 2 (d) 3

2. The interval in which $y = -x^3 + 3x^2 + 2022$ is increasing is

(a) $(-\infty, 0) \cup (2, \infty)$ (b) $(2, \infty)$

(c) $(0, 2)$ (d) $(-\infty, 0)$

3. The maximum value of the function $f(x) = 4\sin x.\cos x$ is

(a) 1 (b) 2

(c) 3 (d) 4

4. Which of the following function is decreasing on $\left(0, \frac{\pi}{2}\right)$

(a) $\cos x$ (b) $\sin x$

(c) $\tan x$ (d) $\sin 2x$

5. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. The rate at which the length of his shadow increases is

(a) 5 km/hr (b) 2 km/hr

(c) 3 km/hr (d) 2.5 km/hr

Answers

ONE MARK QUESTIONS

1. $\frac{\pi}{3}$
2. $196\pi \frac{cm^3}{cm}$
3. $[0, \pi]$
4. $3, \frac{1}{3}$
5. $\frac{1}{e}$
6. $2\sqrt{ab}$
7. $(-\infty, 0)$
8. $a \leq 0$
9. $-\sqrt{2}$
11. 0
12. $\frac{\pi}{2}$
13. –1
14. 0 unit
15. $\frac{1}{2\sqrt{\pi}}$ unit

TWO MARKS QUESTIONS

1. $\frac{1}{2}$
2. 89
3. $\frac{1}{\pi}$ units
4. $10\sqrt{3}\ cm^2 / s$
5. 2a%
6. 3k %
7. ½
8. (2, 4)
10. (0, 0)
11. $\simeq$ 4.042
12. sin 1, – sin 1
13. Local maxima at x = 1
    Local minima at x = 6
14. a = 2, b = –½
15. π cm^2

THREE MARKS QUESTIONS

1. $\frac{1}{\pi}\ cm / s$
2. $\frac{3}{8\pi}\ cm / min$
4. 8 m/sec.
5. 3000 L/s
6. 3 km/h
7. $\frac{35}{88}$ m/h
9. $\frac{6}{49\pi}$ m/min.
10. $1 - 3x + 2x^2$
11. 8 cm^2/sec
13. 25
14. Increasing

15. Increasing for all $x \geq 1$
Decreasing for all $x \leq 1$

16. Increasing on (o, e)
Decreasing on $[e, \infty)$

17. Increasing on
$$\left(O, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$$
Decreasing on $[\frac{3\pi}{4}.\frac{7\pi}{4}]$

20. $(-\infty, 1] U [2, 3]$

21. $[1,\infty]$

23. increasing on $[0, \infty)$
Decreasing $(-\infty, 0]$

24. (i) Strictly increasing
$[-2,1] U [3, \infty)$
(ii) Strictly decreasing
$(-\infty, -2] U [1, 3]$

26. decrease 72 units/sec.

27. $\pi\ cm^2$

28. 0.3%

29. max. value $=\frac{3}{4}$, mim value $=\frac{1}{2}$

30. ab. Max.$= \frac{157}{8}$, ab. Min. $= \frac{-7}{4}$

31. max.value=0,
min.value $= \frac{-3}{5}\left[\frac{2}{5}\right]^{2/3}$

32. ab. Max = 14 at x = 9
ab. Min.$= \frac{-3}{4^{4/3}}$ at $x = \frac{5}{4}$

33. π

FIVE MARKS QUESTIONS

4. $18r^3 + \frac{4}{3}\pi r^3$

9. (1, 2)

10. $\frac{3\sqrt{2}}{8}$

11. $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

13. 2ab sq. Units.

SELF ASSESSMENT TEST-1

1. (b) 2. (a) 3. (d) 4. (c) 5. (c)

SELF ASSESSMENT TEST-2

1. (c) 2. (c) 3. (b) 4. (a) 5. (d)

CHAPTER 7

INTEGRALS

There are many applications of integration in the field such as Physics, Engineering, Business, Economics etc. One of the important application of integration is finding the profit function of producing a certain number of cars if the marginal cost and revenue function are known. Companies can thus determine the maximum profit that can be earned and in this way plan their production, labour and other infrastructure accordingly.

INTEGRALS

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

- Integration as inverse process of differentiation
- Integration of a variety of functions by substitution, by partial fractions and by parts
- Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{px + q}{ax^2 \pm bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int \sqrt{a^2 \pm x^2} dx, \int \sqrt{x^2 - a^2}\, dx$$

$$\int \sqrt{ax^2 + bx + c}\; dx$$

Fundamental Theorem of Calculus (without proof).

- Basic properties of definite integrals and evaluation of definite integrals.

POINTS TO REMEMBER

- Integration or anti derivative is the reverse process of Differentiation.
- Let $\frac{d}{dx}F(x) = f(x)$ then we write $\int f(x)\,dx = F(x) + c$.
- These integrals are called indefinite integrals and c is called constant of integration.
- From geometrical point of view, an indefinite integral is the collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along y-axis.

STANDARD FORMULAE

1. $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c, & n \neq -1 \\ \log_e|x| + c, & n = -1 \end{cases}$

2. $\int (ax+b)^n\,dx = \begin{cases} \frac{(ax+b)^{n+1}}{(n+1)a} + c, & n \neq -1 \\ \frac{1}{a}\log|ax+b| + c, & n = -1 \end{cases}$

3. $\int \sin x\,dx = -\cos x + c.$

4. $\int \cos x\,dx = \sin x + c$

5. $\int \tan x.\,dx = -\log|\cos x| + c = \log|\sec x| + c.$

6. $\int \cot x\,dx = \log|\sin x| + c.$

7. $\int \sec^2 x\,dx = \tan x + c$

8. $\int \mathrm{cosec}^2\,x\,dx = -\cot x + c$

9. $\int \sec x\,\tan x\,dx = \sec x + c$

10. $\int \mathrm{cosec}\,x\,\cot x\,dx = -\mathrm{cosec}\,x + c$

11. $\int \sec x \, dx = \log|\sec x + \tan x| + c$

$$= \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$$

12. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

$$= \log\left|\tan\frac{x}{2}\right| + c$$

13. $\int e^x \, dx = e^x + c$

14. $\int a^x \, dx = \frac{a^x}{\log a} + c$

15. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c \, , |x| < 1$

$$= -\cos^{-1} x + c$$

16. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$$= -\cot^{-1} x + c$$

17. $\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + c \, , |x| > 1$

$$= -\operatorname{cosec}^{-1} x + c$$

18. $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

19. $\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

20. $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

21. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$

22. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

23. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

24. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

25. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$

26. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

RULES OF INTEGRATION

1. $\int [(f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_x(x)]\, dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \ldots\ldots \pm \int f_x(x)dx$

2. $\int k.f(x)dx = k\int f(x)dx.$

3. $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + c$

INTEGRATION BY SUBSTITUTION

1. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

2. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$

3. $\int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{-n+1}}{-n+1} + c$

INTEGRATION BY PARTS

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int\left[f'(x)\int g(x)\, dx\right]$$

DEFINITE INTEGRALS

$$\int_a^b f(x)dx = F(b) - F(a), \text{where } F(x) = \int f(x)\, dx$$

DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x)dx = \lim_{h\to o} h\,[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+n-1h)]$$

$Where\ h = \frac{b-a}{h}$ $or \int_a^b f(x)dx = \lim_{h\to o}[h \sum_{r=1}^{n} f(a+rh)]$

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x) = -\int_b^a f(x)dx$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

4. $(i) \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

(ii) $\int_o^a f(x)\,dx = \int_o^a f(a-x)dx$

5. $\int_{-a}^{a} f(x)dx = 2\int_o^a f(x)dx,\quad if\ f\ (x)\ is\ even\ function$

6. $\int_{-a}^{a} f(x)dx = 0\ if\ f\ (x)\ is\ an\ odd\ function$

7. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

Illustration:

Evaluate $\int e^x \left(\frac{x+2}{x+4}\right)^2 dx$

Solution: $I = \int e^x \left(\frac{x+2}{x+4}\right)^2 dx = \int e^x \left(1 - \frac{2}{x+4}\right)^2 dx$

$$= \int e^x \left[\left(1 - \frac{4}{x+4}\right) + \frac{4}{(x+4)^2}\right] dx$$

$$= \int e^x \left[f(x) + f'(x)\right] dx, \text{ where } f(x) = 1 - \frac{4}{x+4}$$

$$= e^x f(x) + C = e^x \left(1 - \frac{4}{x+4}\right) + C = \frac{xe^x}{x+4} + C$$

Illustration:

Find $\int \frac{x^2+1}{(x+1)^2} dx$

Solution: $\int \frac{x^2+1}{(x+1)^2} dx = \int \frac{(x+1)^2 - 2x}{(x+1)^2} dx$

$$= \int \frac{(x+1)^2 - 2(4+1) + 2}{(x+1)^2} dx$$

$$= \int \left[1 - \frac{2}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= x - 1 \log|4+1| - \frac{2}{x+1} + C$$

Illustration:

Evaluate $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$

Solution: $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \int_0^{\pi/4} \frac{\tan^2 x \sec^2 x}{(\tan^3 x + 1)^2} dx$

[dividing Num and Den by $\cos^6 x$]

Put $z = \tan^3 x + 1$,

then $d2 = 3\tan^2 x \sec^2 x \, dx$

Also when $x = 0$, $z = 0$ and when $x = \frac{\pi}{4}$, $z = 2$

Now $I = \frac{1}{3}\int_2^1 \frac{dz}{z^2} = -\frac{1}{3}\left[\frac{1}{z}\right]_1^2 = -\frac{1}{3}\left[\frac{1}{2} - 1\right] = \frac{1}{6}$

Illustration:

Find $\int \frac{x^2+1}{(x+1)^2} dx$

Solution: $\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx = \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x} dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x} dx$

$= 0 + \frac{\pi}{4}.2\int_0^{\pi/4} \frac{dx}{2 - \cos x}$ [Since first functionis an even function and second function is an odd function

$= \frac{\pi}{2}\int_0^{\pi/4} \frac{dx}{2(1 - 2\sin^2 x)}$

$= \frac{\pi}{2}\int_0^{\pi/4} \frac{dx}{2\sin^2 x + 1}$

$= \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x + 1} dx$ [dividing num and den by $\cos^2 x$]

Put $z = \sqrt{3}\tan x$, then $dz = \sqrt{3}\sec^2 x\, dx$

Also when $x = 0$, $z = 0$, and when $x = \frac{\pi}{4}$, $z = \sqrt{3}$

$\therefore$ From (i), $I = \frac{\pi}{2}\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{dz}{z^2 + 1} = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1} z\right]_0^{\sqrt{3}}$

$= \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1} 0\right]$

$= \frac{\pi}{2\sqrt{3}}\tan^{-1}\sqrt{3}$

$= \frac{\pi}{2\sqrt{3}}.\frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$

ONE MARK QUESTIONS

Evaluate the following integrals:

1. $\int \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}\right)\, dx$

2. $\int_{-1}^{1} e^{|x|}\, dx$

3. $\int \frac{dx}{1-\sin^2 x}$

4. $\int_{-1}^{1} x^{99} \cos^4 x\, dx$

5. $\int \frac{1}{x\ \log x \log(\log x)} dx$

6. $\int_{-1/2}^{1/2} \cos x . \log\left(\frac{1+x}{1-x}\right) dx$

7. $\int \left(e^{a \log x} + e^{x \log a}\right) dx$

8. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

9. $\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx$

10. $\int \sqrt{10 - 4x + x^2}\ dx$

11. $\int_{-1}^{1} x^3 |x|\, dx$

12. $\int \frac{1}{\sin^2 x\ \cos^2 x}\ dx$

13. $\int_{-2}^{2} \frac{dx}{1+|x-1|}$

14. $\int e^{-\log x}\, dx$

15. $\int \frac{e^x}{a^x}\, dx$

16. $\int \frac{x}{\sqrt{x+1}} dx$

17. $\int \frac{x}{(x+1)^2}\,dx$

18. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

19. $\int \cos^2 \alpha\,dx$

20. $\int \frac{1}{x\cos\alpha + 1}\,dx$

21. $\int \sec x \log(\sec x + \tan x)\,dx$

22. $\int \frac{1}{\cos\alpha + x\ \sin\alpha}dx$

23. $\int \frac{\sec^2(\log x)}{x}\,dx$

24. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$

25. $\int \frac{1}{x(2+3\ \log x)}\,dx$

26. $\int \frac{1-\sin x}{x+\cos x}dx$

27. $\int \frac{1-\cos x}{\sin x}\,dx$

28. $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}\,dx$

29. $\int \frac{(x+1)}{x}(x+\log x)\,dx$

30. $\int_0^{\pi} |\cos x|\,dx$

31. $\int_0^2 [x]\,dx$ *where* $[x]$*is greatest integers function.*

32. $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

33. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$

34. $\int_{-2}^{1} \frac{|x|}{x}\,dx$

35. $\int_{-1}^{1} x\,|x|\,dx$

36. $\int x\sqrt{x+2}\,dx$

37. $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx$

38. $\int \frac{\sin x}{\sin 2x}\,dx$

39. $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx$

40. $\int \frac{1}{\sec x + \tan x}\,dx$

41. $\int \frac{\sin^2 x}{1+\cos x}\,dx$

42. $\int \frac{1-\tan x}{1+\tan x}\,dx$

TWO MARKS QUESTIONS

Evaluate :

1. $\int e^{[\log(x+1)-\log x]}dx$

2. $\int \frac{1}{\sqrt{x+1}+\sqrt{x+2}}dx$

3. $\int \sin x \sin 2x\,dx$

4. $\int \left[\frac{x}{a}+\frac{a}{x}+x^a+a^x\right]dx$

5. $\int_0^{\pi/2} \log\left(\frac{5+3\cos x}{5+3\sin x}\right)dx$

6. $\int \frac{a^x+b^x}{c^x}dx$

7. $\int \left(\sqrt{ax}-\frac{1}{\sqrt{ax}}\right)^2 dx$

8. $\int e^x 2^x\,dx$

9. $\int 2^{2^{2^x}} 2^{2^x} 2^x\,dx$

10. $\int \frac{\sin\left(2\tan^{-1}x\right)}{1+x^2}dx$

11. $\int x\log 2x\,dx$

12. $\int_0^{\pi/4} \sqrt{1+\sin 2x}dx$

13. $\int_0^{\pi/2} e^x\left(\sin x-\cos x\right)dx$

14. $\int_4^9 \frac{\sqrt{x}}{\left(30-x^{3/2}\right)}dx$

15. $\int_0^1 \frac{dx}{e^x+e^{-x}}dx$

16. $\int \frac{\log|\sin x|}{\tan x}dx$

17. $\int \frac{\sin^4 x+\cos^4 x}{\sin^3 x+\cos^3 x}dx$

18. $\int \sqrt{\tan x}\,(1+\tan^2 x)dx$

19. $\int \frac{\sin 2x}{(a+b\cos x)^2}dx$

20. $\int \frac{x^2-x+2}{x^2+1}dx$

THREE MARKS QUESTIONS

Evaluate :

1. (i) $\int \frac{x\, cosec\,(\tan^{-1} x^2)}{1+x^4}\, dx$

(ii) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\, dx$

(iii) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

(iv) $\int \frac{\cos(x+a)}{\cos(x-a)}\, dx$

(v) $\int \cos 2x\ \cos 4x\ \cos 6x\ dx$

(vi) $\int \tan 2x \tan 3x \tan 5x\, dx$

(vii) $\int \sin^2 x \cos^4 x\ dx$

(viii) $\int \cot^3 x\, cosec^4 x\, dx$

(ix) $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$ [Hint: Put $a^2 \sin^2 x + b^2 \cos^2 x = t$ or t^2

(x) $\int \frac{1}{\sqrt{\cos^3 x\ \cos(x+a)}}\, dx$

(xi) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$

(xii) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

Evaluate :

(i) $\int \frac{x}{x^4+x^2+1} dx$

(ii) $\int \frac{1}{x[6(\log x)^2+7\log x+2]} dx$

(iii) $\int \frac{1}{\sqrt{\sin^3 x \cos^5 x}} dx$

(iv) $\int \frac{x^2+1}{x^4+1} dx$

(v) $\int \frac{1}{\sqrt{(x-a)(x-b)}} dx$

(vi) $\int \frac{5x-2}{3x^2+2x+1} dx$

(vii) $\int \frac{x^2}{x^2+6x+1} dx$

(viii) $\int \frac{x+2}{\sqrt{4x-x^2}} dx$

(ix) $\int x\sqrt{1+x-x^2} dx$

(x) $\int \frac{\sin^4 x}{\cos^8 x} dx$

(xi) $\int \sqrt{\sec x - 1} dx$ [Hint: Multiply and divided by $\sqrt{\sec x + 1}$]

Evaluate :

3. (i) $\int \frac{dx}{x(x^7+1)}$

(ii) $\int \frac{3x+5}{x^3-x^2-x+1} dx$

(iii) $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}\, d\theta$

(iv) $\int \frac{dx}{(2-x)(x^2+3)}$

(v) $\int \frac{x^2+x+2}{(x-2)(x-1)}\, dx$

(vi) $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$

(vii) $\int \frac{dx}{(2x+1)(x^2+4)}$

(viii) $\int \frac{x^2-1}{x^4+x^2+1} dx$

(ix) $\int \sqrt{\tan x}\, dx$

(x) $\int \frac{dx}{\sin x - \sin 2x}$

4. Evaluate:

(i) $\int x^5 \sin x^3\, dx$

(ii) $\int \sec^3 x\, dx$

(iii) $\int e^{ax} \cos(bx + c)\, dx$

(iv) $\int \sin^{-1}\left(\frac{6x}{1+9x^2}\right) dx$ [Hint: Put $3x = \tan\theta$]

(v) $\int \cos\sqrt{x}\, dx$

(vi) $\int x^3 \tan^{-1} x\, dx$

(vii) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

(viii) $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

(ix) $\int \sqrt{2ax - x^2}\, dx$

(x) $\int e^x \frac{(x^2+1)}{(x+1)^2} dx$

(xi) $\int x^3 \sin^{-1}\left(\frac{1}{x}\right) dx$

(xii) $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\} dx$ [Hint: Put $\log x = t$, $x = e^t$]

(xiii) $\int (6x+5)\sqrt{6 + x - x^2}\, dx$

(xiv) $\int \frac{1}{x^3+1} dx$

(xv) $\int \tan^{-1}\left(\frac{x-5}{1+5x}\right) dx$

(xvi) $\int \frac{dx}{5+4\cos x}$

5. Evaluate the following definite integrals:

(i) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

(ii) $\int_0^{\pi/2} \cos 2x \, \log \sin x \, dx$

(iii) $\int_0^1 x \sqrt{\frac{1-x^2}{1+x^2}}\, dx$

(iv) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

(v) $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

(vi) $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$

(vii) $\int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x}\, dx$

(viii) $\int_0^1 x \log\left(1 + \frac{x}{2}\right) dx$

(ix) $\int_{-1}^{1/2} |x \cos \pi x|\, dx$

(x) $\int_{-\pi}^{\pi} (\cos a\, x - \sin b\, x)^2\, dx$

6. Evaluate:

(i) $\int_2^5 [|x-2| + |x-3| + |x-4|]\, dx$

(ii) $\int_0^{\pi} \frac{x}{1+\sin}\, dx$

(iii) $\int_{-1}^1 e^{\tan^{-1} x}\left[\frac{1+x+x^2}{1+x^2}\right] dx$

(iv) $\int_0^{\pi} \frac{x\ \sin x}{1+\cos^2 x}\, dx$

(v) $\int_0^2 [x^2]\, dx$

(vi) $\int_0^{\pi/2} \frac{x \sin x\ \cos x}{\sin^4 x + \cos^4 x}\, dx$

(vii) $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ [*Hint: use* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

7. Evaluate the following integrals:

(i) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

(ii) $\int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)\,dx$

(iii) $\int_{0}^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$

(iv) $\int_{0}^{\pi} \frac{x \tan x}{\sec x + \operatorname{cosec} x}\,dx$

(v) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$

8. Evaluate

(*i*) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx \quad x \in [0, 1]$

(*ii*) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

(iii) $\int \frac{x^2 e^x}{(x+z)^2}\,dx$

(iv) $\int \frac{x^2}{(x \sin x + \cos x)^2}\,dx$

(v) $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\,dx$

(vi) $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} \, dx$

(vii) $\int \frac{\sin x}{\sin 4x} \, dx$

(viii) $\int_{-1}^{3/2} |x \sin \pi x| \, dx$

(ix) $\int \frac{\sin(x-a)}{\sin(x+a)} \, dx$

(x) $\int \frac{x^2}{(x^2+4)(x^2+9)} \, dx$

(xi) $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \, dx$

FIVE MARKS QUESTIONS

9. Evaluate the following integrals:

(i) $\int \frac{x^5+4}{x^5-x} \, dx$

(ii) $\int \frac{2e^t}{e^{3t} - 6e^{2t} + 11e^t - 6} \, dt$

(iii) $\int \frac{2x^3}{(x+1)(x-3)^2} \, dx$

(iv) $\int \frac{1+\sin x}{\sin x\,(1+\cos x)} \, dx$

(v) $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

(vi) $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

(vii) $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

10. Evaluate the following integrals as limit of sums:

(i) $\int_2^4 (2x+1)\,dx$

(ii) $\int_0^2 (x^2+3)\,dx$

(iii) $\int_1^3 (3x^2-2x+4)\,dx$

(iv) $\int_0^4 (3x^2+e^{2x})\,dx$

(v) $\int_0^1 e^{2-3x}\,dx$

(vi) $\int_0^1 (3x^2+2x+1)\,dx$

11. Evaluate:

(i) $\int \frac{dx}{(\sin x-2\cos x)(2\sin x+\cos x)}$

(ii) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

(iii) $\int_0^{\pi/2} (2\log\sin x-\log\sin 2x)\,dx$

12. $\int_0^1 x(\tan^{-1} x)^2 \; dx$

13. $\int_0^{\pi/2} \log \sin x \; dx$

14. $Prove\ that \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx = 2 \int_0^1 \tan^{-1} x \, dx$

Hence or otherwise evaluate the integral $\int \tan^{-1}(1 - x + x^2)\, dx$.

15. $Evaluate \int_0^{\pi/2} \frac{sin^2 x}{\sin x + \cos x} dx$.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. $I = \int (x^8 + 1)(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)dx =$

   (a) $x^{16} - 1 + c$ (b) $x^{17} - x + c$

   (c) $\frac{x^{17}}{17} - x + c$ (d) $\frac{x^{16}}{16} - x + c$

2. $\int \sin(x^2 + 2022)\, d(x^2) =$

   (a) $2x.\sin(x^2 + 2022) + c$ (b) $-2x.\cos(x^2 + 2022) + c$

   (c) $\sin(x^2 + 2022) + c$ (d) $-\cos(x^2 + 2022) + c$

3. $\int \cos^3 x.\sqrt{\sin x}\, dx = \frac{2\sin^a x}{3} - \frac{2\sin^b x}{7} + c$, then $(a + b) =$

   (a) 2 (b) 4

   (c) 5 (d) 6

4. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x.\cos^2 x} dx =$

   (a) $\tan x + \cot x + c$ (b) $\tan x - \cot x + c$

5. $\int_{0}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{k}$, then $k =$

(a) 0.25 (b) 0.5

(c) 1 (d) 4

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. $\int_{0}^{\pi/2} \log \tan x\,dx =$

(a) 0 (b) 1

(c) π (d) $\frac{\pi}{2}$

2. $\int_{0}^{\pi} \frac{x}{1+\sin x} dx =$

(a) 4π (b) $\frac{\pi}{2}$

(c) π (d) 2π

3. $\int \log(x^2+1) dx =$

(a) $x\log(x^2+1)-2x+2\tan^{-1}x+c$ (b) $x\log(x^2+1)-2x-2\tan^{-1}x+c$

(c) $x\log(x^2+1)+2x+2\tan^{-1}x+c$ (d) None of these

4. $\int e^x.\sin x\,dx =$

(a) $\frac{e^x(\sin x+\cos x)}{2}+c$ (b) $\frac{e^x(\sin x-\cos x)}{2}+c$

(c) $\frac{e^x(-\sin x+\cos x)}{2}+c$ (d) $\frac{-e^x(\sin x+\cos x)}{2}+c$

5. $\int \cos^2 x\,dx = ax + b\sin 2x + c$, then $(2a+4b+1) =$

(a) 0 (b) 1

(c) 3 (d) –7

Answers

ONE MARKS QUESTIONS

1. $\frac{\pi}{2}x + c$
2. $2e - 2$
3. $\tan x + c$
4. 0
5. $\log|\log|\log x|| + c$
6. 0
7. $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$
8. $\tan x + c$
9. 0
10. $\frac{(x-2)\sqrt{x^2-4x+10}}{2} + 3\log\left|(x-2) + \sqrt{x^2-4x+10}\right| + c$
11. 0
12. $\tan x - \cot x + c$
13. $3\log_e 2$
14. $\log|x| + c$
15. $\frac{\left(\frac{e}{a}\right)^x}{\log\left(\frac{e}{a}\right)} + c$
16. $\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + c$
17. $\log|x+1| + \frac{1}{x+1} + c$
18. $2e^{\sqrt{x}} + c$
19. $x\cos^2\alpha + c$
20. $\frac{\log|x\ \cos\alpha+1|}{\cos\alpha} + c$
21. $\frac{(\log|\sec x+\tan x|)^2}{2} + c$
22. $\frac{\log|\cos\alpha+x\sin\alpha|}{\sin\alpha} + c$
23. $\tan|\log x| + c$
24. $\log\left|e^x + \sqrt{4+e^{2x}}\right| + c$
25. $\frac{1}{3}\log|2 + 3\log x| + c$
26. $\log|x + \cos x| + c$
27. $2\log\left|\sec\frac{x}{2}\right| + c$
28. $\frac{1}{e}\log|x^e + e^x| + c$
29. $\frac{(x+\log x)^2}{2} + c$
30. 2
31. 1

32. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

33. $\frac{b-a}{2}$

34. –1

35. 0

36. $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+c$

37. 0

38. $\frac{1}{2}\log|\sec x+\tan x|+c$

39. 2-$\sqrt{2}$

40. $\log|1+\sin x|+c$

41. $x-\sin x+c$

42. $\log|\cos x+\sin x|+c$

TWO MARKS QUESTIONS

1. x + log x + c

2. $\frac{2}{3}\left[(x+2)^{3/2}-(x+1)^{3/2}\right]+c$

3. $\frac{-1}{2}\left[\frac{\sin 3x}{3}-\sin x\right]+c$

4. $\frac{1}{a}\frac{x^2}{2}+a\log|x|+\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$

5. 0

6. $\frac{\left(\frac{a}{c}\right)^x}{\log\left|\frac{a}{c}\right|}+\frac{\left(\frac{b}{c}\right)^x}{\log\left|\frac{b}{c}\right|}+c$

7. $\frac{ax^2}{2}+\frac{\log|x|}{a}-2x+c$

8. $\frac{2^x e^x}{\log(2e)}+c$

9. $\frac{2^{2^{2^x}}}{(\log 2)^3}+C$

10. $\frac{-\left[\cos 2\left(\tan^{-1}x\right)\right]}{2}+C$

11. $\frac{x^2}{2}\log 2x-\frac{x^2}{4}+C$

12. 1

13. 1

14. $\frac{19}{99}$

15. $\tan^{-1}e-\frac{\pi}{4}$

16. $\frac{\log|\sin x|^2}{2}+C$

17. log|secx + tan x|
+ log |cosec x – cot x| + C

18. $\frac{2}{3}(\tan x)^{3/2}+C$

19. $-\frac{2}{b^2}\left[\log|a+b\cos x|+\frac{a}{a+b\cos x}\right]+C$

20. $x-\frac{1}{2}\log\left|x^2+1\right|+\tan^{-1}x+C$

THREE MARKS QUESTIONS

1. (i) $\frac{1}{2}\log\left[\text{cosec}(\tan^{-1}x^2) - \frac{1}{x^2}\right] + c$

(ii) $\frac{1}{2}\left(x^2 - x\sqrt{x^2-1}\right) + \frac{1}{2}\log\left|x + \sqrt{x^2-1}\right| + c$

(iii) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$

(iv) $x\cos 2a - \sin 2a\ \log|\sec(x-a)| + c$

(v) $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

(vi) $\frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| + c$

(vii) $\frac{1}{32}\left[2x + \frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x - \frac{1}{6}\sin 6x\right] + c$

(viii) $-\left(\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4}\right) + c$

(ix) $\frac{1}{a^2-b^2}\sqrt{a^2\sin^2 x + b^2\cos^2 x} + c$

(x) $-2\text{cosec}\ a\sqrt{\cos a - \tan x \sin a} + c$

(xii) $\tan x - \cot x - 3x + c$

(vi) $\sin^{-1}[\sin x - \cos x] + c$

2. (i) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$

(ii) $\log\left|\frac{2\log x}{3\log x}\right| + c$

(iii) $\frac{-2}{\sqrt{\tan x}} + \frac{2}{3}\tan^{3/2}x + c$

(iv) $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right\}+c$

(v) $2\log\left|\sqrt{x-a}+\sqrt{x-b}\right|+c$

(vi) $\frac{5}{6}\log|3x^2+2x+1|+\frac{-11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c$

(vii) $x-3\log|x^2+6x+12|+2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)+c$

(viii) $-\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+c$

(ix) $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

(x) $\frac{\tan^5 x}{5}+\frac{\tan^7 x}{7}+c$

(xi) $-\log\left|\cos x+\frac{1}{2}+\sqrt{\cos^2 x+\cos x}\right|+c$

3. (i) $\frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right|+c$

(ii) $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+c$

(iii) $\frac{-2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|1+\cos\theta|+c$

(iv) $\frac{1}{14}\log\left|\frac{x^2+3}{(2-x)^2}\right|+\frac{2}{7\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

(v) $x+4\log\left|\frac{(x-2)^2}{x-1}\right|+c$

(vi) $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+c$

(vii) $\frac{2}{17}\log|2x+1| - \frac{1}{17}\log|x^2+4| + \frac{1}{34}\tan^{-1}\frac{x}{2} + c$

(viii) $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

(ix) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right| + c$

(x) $-\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1 - 2\cos x| + c$

4. (i) $\frac{1}{3}[-x^3\cos x^3 + \sin x^3] + c$

(ii) $\frac{1}{2}[\sec x\tan x + \log|\sec x + \tan x|] + c$

(iii) $\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + c$

(iv) $2x\ \tan^{-1}3x - \frac{1}{3}\log|1+9x^2| + c$

(v) $2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

(vi) $\left(\frac{x^4-1}{4}\right)\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c$

(vii) $\frac{1}{2}e^{2x}\tan x + c$

(viii) $\frac{x}{\log x} + c$

(ix) $\left(\frac{x-a}{2}\right)\sqrt{2ax - x^x} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$

(x) $e^x\left(\frac{x-1}{x+1}\right) + c$

(xi) $\frac{x^4}{4}\sin^{-1}\left(\frac{1}{x}\right) + \frac{x^2+2}{12}\sqrt{x^2-1} + c$

(xii) $x\log|\log x| - \frac{x}{\log x} + c$

(xiii) $-2(6+x-x^2)^{\frac{3}{2}} + 8\left[\frac{2x-1}{4}\sqrt{6+x-x^2} + \frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right)\right] + c$

(xiv) $\frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2 - x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$

(xv) $x\tan^{-1}x - \frac{1}{2}\log|1 + x^2| - x\tan^{-1}5 + c$

(xvi) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$

5. (i) $\frac{1}{20}\log 3$

(ii) $-\pi/4$

(iii) $\frac{\pi}{4} - \frac{1}{2}$

(iv) $\frac{\pi}{4} - \frac{1}{2}\log 2$

(v) $\frac{\pi}{2}$

(vi) $\pi/4$

(vii) $\pi/2$

(viii) $\frac{3}{4} + \frac{3}{2}\log\frac{2}{3}$

(ix) $\frac{3}{2\pi} - \frac{1}{\pi^2}$

(x) $2\pi + \frac{1}{2a}\sin 2a\pi - \frac{1}{2b}\sin 2b\pi$

6. (i) $\frac{1}{2}$

(ii) π

(iii) $e^{\pi/4} + e^{-\pi/4}$

(iv) $\frac{1}{4}\pi^2$

(v) $5 - \sqrt{3} - \sqrt{2}$

(vi) $\frac{\pi^2}{16}$ (vii) $\frac{\pi^2}{2a}$

7. (i) $\frac{\pi}{12}$ (ii) 2

(iii) $\frac{\pi}{2}$ (iv) $\frac{\pi^2}{4}$

(v) $a\pi$

8. (i) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x-x^2}}{\pi}-x+c$

(ii) $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+c$

(iii) $\frac{x-2}{x+2}e^x+c$

(iv) $\frac{\sin x-x\cos x}{x\sin x+\cos x}+c$

(v) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+c$

(vi) $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

(vii) $\frac{1}{8}\log\left|\frac{1-\sin x}{1+\sin x}\right|-\frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+c$

(viii) $\frac{3}{\pi}+\frac{1}{\pi^2}$

(ix) $(\cos 2a)(x+a)-(\sin 2a)\log|\sin(x+a)|+c$

(x) $-\frac{4}{5}\log|x^2+4|+\frac{9}{5}\log|x^2+9|+c$

(xi) $-\left(\frac{1}{2}\sin 2x+\sin x\right)+c$

9. (i) $x-4\log|x|+\frac{5}{4}\log|x-1|+\frac{3}{4}\log|x+1|+\log|x^2+$

(ii) $1|\frac{-1}{2}\tan^{-1}x+c$

(iii) $\log\left|\frac{(e^t-1)(e^t-3)}{(e^t-2)^2}\right|+c$

(iv) $2x-\frac{1}{8}\log|x+1|+\frac{81}{8}\log|x-3|-\frac{27}{2(x-3)}+c$

(v) $\frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos}\right|+\frac{1}{2(1+\cos\ \ )}+\tan\frac{x}{2}+c$

(vi) $\frac{\pi}{\sqrt{2}}$ (vii) $\frac{\pi-2}{4}$

(viii) $\frac{\pi}{4}-\frac{1}{2}\log 2$

10. (i) 14 (ii) $\frac{26}{3}$

(iii) 26 (v) $\frac{1}{3}\left(e^2-\frac{1}{e}\right)$

(iv) $\frac{1}{2}(127+e^8)$ (vi) 3

11. (i) $\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right|+c$ (ii) $\frac{\pi}{8}\log 2$

(iii) $\frac{\pi}{2}\log\frac{1}{2}$

12. $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$

13. $\frac{-\pi}{2}\log 2$

14. $\log 2$

15. $\frac{1}{\sqrt{2}}\log\left|\sqrt{2}+1\right|$

SELF ASSESSMENT TEST-1

1. (c) 2. (d) 3. (c) 4. (a) 5. (d)

SELF ASSESSMENT TEST-2

1. (a) 2. (c) 3. (a) 4. (b) 5. (c)

CHAPTER 8

APPLICATION OF INTEGRALS

In real life, integrations are used in various fields such as engineering, where engineers use integrals to find the shape of building. In Physics, used in the centre of gravity etc. In the field of graphical representation. Where three-dimensional models are demonstrated.

The PETRONAS TOWERS in KUALA LUMPUR experience high forces due to wind. Integration was used to create this design of building.

APPLICATION OF INTEGRALS

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

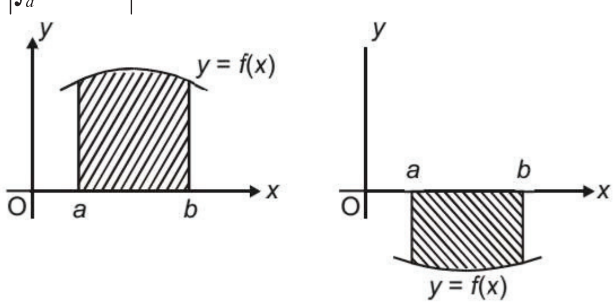
- Application in finding the area under simple curves, especially lines, circles/ parabolas/ellipse (in standard form only)

POINTS TO REMEMBER

AREA OF BOUNDED REGION

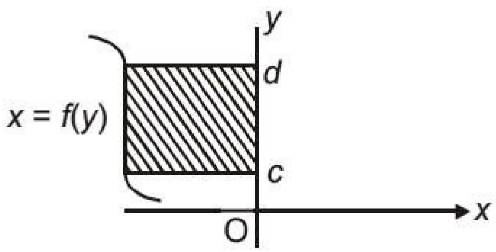
- Area bounded by the curve $y = f(x)$, the x axis and between the ordinates, $x = a$ and $x = b$ is given by

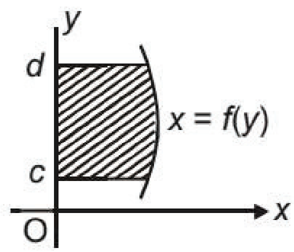
  $$Area = \left|\int_a^b f(x)dx\right|$$



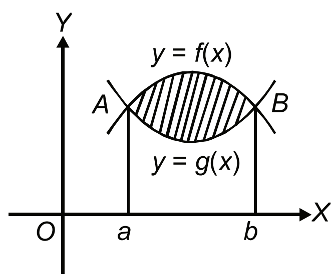
- Area bounded by the curve $x = f(y)$, the y-axis and between the abscissas, y = c and y = d is given by

$$Area = \left|\int_c^d f(y)dy\right|$$





- Area bounded by two curves $y = f(x)$ and $y = g(x)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in [a, b]$ and between the ordinates $x = a$ and $x = b$ is given by



Area = $\int_a^b [f(x) - g(x)]dx$

- Area of the following shaded region = $\left|\int_a^k f(x)dx\right| + \int_k^b f(x)dx$

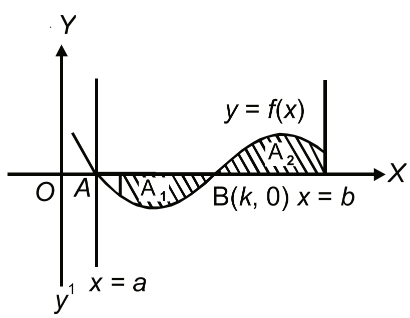


Illustration:

Using integration. Find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 2$ and $x = 4$

Solution: Required area= Areaof PQRS

= Area bounded by the line $2y + x = 8$, x-axis and ordinates $x = 2$, $x = 4$

$$= \int_2^4 y\,dx = \int_2^4 \frac{8-x}{2}\,dx$$

$$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4 = \frac{1}{2}[(32-8)-(16-2)]$$

$$= \frac{1}{2}[24-14] = \frac{1}{2}\times 10 = 5\,\text{sq.units}$$

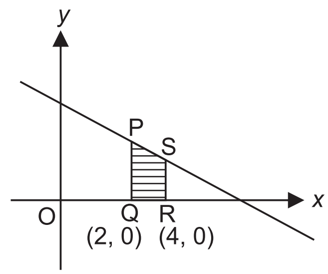


Illustration:

Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x-varies from 0 to $\pi/2$. Find the area of the region enclosed by the curves and the x-axis.

Solution: Given curves $y = \sin x$

and $y = \cos x$

Area of shaded region

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= -[\cos x]_0^{\pi/2} + [\sin x]_{\pi/4}^{\pi/2} = -\left[\frac{1}{\sqrt{2}} - 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{-1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}} = (2 - 2\sqrt{2}) \quad \text{square units}$$

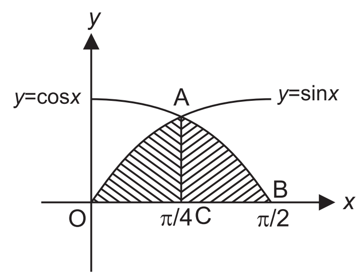


Illustration:

Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.

Solution: Given curve $y^2 = 16x$

line $x = 4$

Area of shaded region

= 2(area of AOC)

$$= 2\int_0^4 y\,dx = 2\int_0^4 4\sqrt{x}\,dx$$

$$= 8 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 = \frac{16}{3}[8] = \frac{128}{3}\ \text{sq.units}$$

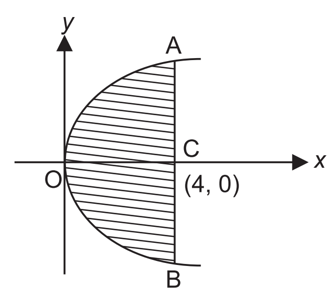


Illustration:

Using integration, find the area of the smaller portion of the circle $x^2 + y^2 = 4$ cut off by the line $x = 1$.

Solution: Circle $x^2 + y^2 = 4$

line $x = 1$

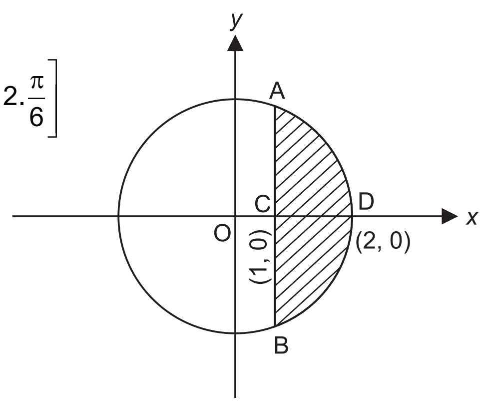
Area of shaded region

= 2(area bounded by the circle, the x-axis and ordinate $x = 1$ and $x = 2$)

$$= 2\int_1^2 y\,dx = 2\int_1^2 \sqrt{4 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 = 2\left[2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{6}\right]$$

$$= 2\left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right] = \frac{4\pi}{3} - \sqrt{3} \text{ sq. units}$$



ONE MARK QUESTIONS

1. Find the area bounded by $y = \sin 2x$, $0 \le x \le \frac{\pi}{4}$ and the coordinate axes.
2. Find the area bounded by $y = \cos 3x$, $0 \le x \le \frac{\pi}{6}$ and the coordinate axes.
3. Find the area bounded by the line $x + 2y = 8$, x – axis and the lines $x = 1$ and $x = 3$.
4. Find the area bounded by the curve $y = x^3$, x – axis and the lines $x = 0$ and $x = 4$.
5. Find the area of region bounded by the curve $y = x^2$, x – axis and the lines $x = -1$, $x = 1$.

TWO MARKS QUESTIONS

Using Integration:

1. Find the area of the circle $x^2 + y^2 = 16$.
2. Find the area of the parabols $y^2 = 4a\,x$ bounded by its latus rectum.
3. Find the area bounded by the curve $y^2 = x$, x – axis and the lines $x = 0$, $x = 4$.
4. Find the area bounded by the region $\{(x, y): x^2 \le y \le |x|\}$.
5. Find the area bounded by the region $y = 9x^2$, $y = 1$ and $y = 4$.
6. Find the area bounded by the curve $y = \sin x$ between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
7. Find the area bounded by the lines $y = 2x + 3$, $y = 0$, $x = 2$ and $x = 4$.
8. Find the area of the region bounded by $y^2 = 4x$, $x = 1$, $x = 4$ and x-axis in the first quadrant.
9. Find the area bounded by the curves $y^2 = 4ax$ and the lines $y = 2a$ and y-axis.
10. Find the area of the triangle formed by the straight lines $y = 2x$, $x = 0$ and $y = 2$ by integration.

THREE/FIVE MARKS QUESTIONS

Using Integration:

1. Find the area bounded by the curve $4y = 3x^2$ and the line $3x – 2y + 12 = 0$.
2. Find the area bounded by the curve $x = y^2$ and the line $x + y = 2$.
3. Find the area of the triangular region whos verties are (1, 2), (2 – 2) and (4, 3).
4. Find the area bounded by the region $\{(x, y): x^2 + y^2 \le 1 \le x + \frac{y}{2}\}$
5. Find the area of the region bounded by the lines $x – 2y = 1$, $3x – y – 3 = 0$ and $2x + y – 12 = 0$.
6. Prove that the curve $y = x^2$ and, $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$, $y = 1$ into three equal parts.
7. Find the area of the smaller region enclosed between ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the line $bx + ay = ab$.
8. Using integration, find the area of the triangle whose sides are given by $2x + y = 4,\ 3x - 2y = 6\ and\ x - 3y + 5 = 0.$
9. Using integration, find the area of the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

10. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

11. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

12. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x^2 = y$ and y-axis.

13. Using integration, find the area of the region bounded by the curve $y = 1 + |x + 1|$ and lines $x = -3, x = 3, y = 0$.

14. Find the area of the region enclosed between curves $y = |x - 1|$ and $y = 3 - |x|$.

15. If the area bounded by the parabola $y^2 = 16ax$ and the line $y = 4mx$ is $\frac{a^2}{12}$ sq unit then using integration find the value of m.

16. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $y = x$ and x-axis in first quadrant.

17. Find the area bounded by the parabola $y^2 = 4x$ and the straight line $x + y = 3$.

18. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

19. Find the area of region $\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \leq \frac{x}{3} + \frac{y}{2}\right\}$

20. Using integration, find the area of the triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2).

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

   (a) $\frac{9}{2}$ (b) $\frac{9}{3}$

   (c) $\frac{9}{4}$ (d) $\frac{9}{5}$

2. Area lying in first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

   (a) π (b) $\frac{\pi}{3}$

   (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

3. The area of the region bounded by the curve y = x + 1 and the lines x = 2, x = 3 and x-axis is

(a) $\frac{13}{2}$ sq.units (b) $\frac{11}{2}$ sq.units

(c) $\frac{9}{2}$ sq.units (d) $\frac{7}{2}$ sq.units

4. The area bounded by the curve $y^2 = x$ and the line $x = 2y$ is

(a) $\frac{1}{3}$ sq.units (b) $\frac{2}{3}$ sq.units

(c) 1 sq. unit (d) $\frac{4}{3}$ sq.units

5. The area of the region bounded by the $y = \sin x$, $y = \cos x$ and y-axis, $0 \le x \le \frac{\pi}{4}$ is

(a) $(\sqrt{2}+1)$ sq.units (b) $(\sqrt{2}-1)$ sq.units

(c) $2\sqrt{2}$ sq.units (d) $(2\sqrt{2}-1)$ sq.units

ANSWERS

ONE MARKS QUESTION

1. $\frac{1}{2}$ square units.
2. $\frac{1}{2}$ square units.
3. 7 square units.
4. 64 square units.
5. 2/3 square units.

TWO MARKS QUESTIONS

1. 16π square units.
2. $\frac{8}{3}a^2$ square units.
3. $\frac{16}{3}$ square units.
4. $\frac{1}{3}$ square units.
5. $\frac{28}{9}$ square units.
6. 2 square units.
7. 18 square units.
8. $\frac{28}{3}$ square units.
9. $\frac{2}{3}a^2$ square units.
10. 1 square units.

THREE/FIVE MARKS QUESTIONS

1. 27 square units.
2. $\frac{9}{2}$ square units.
3. $\frac{13}{2}$ square units.

4. $\left(\frac{\pi}{4}-\frac{2}{5}-\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$ square units.

5. 10 square units.

7. $\left(\frac{\pi-2}{4}\right)ab$ square units.

8. 3.5 square units.

9. 4 square units.

10. $\left(\pi-\frac{1}{2}\right)$ square units.

11. $\frac{9}{8}$ square units.

12. $\frac{10}{3}$ square units.

13. 16 square units.

14. 4 square units.

15. m = 2.

16. 2π sq. units

17. $\frac{64}{3}$ sq. units

18. 9 sq. units

19. $\frac{3}{2}(\pi-2)$ sq. units

20. 7 sq. units

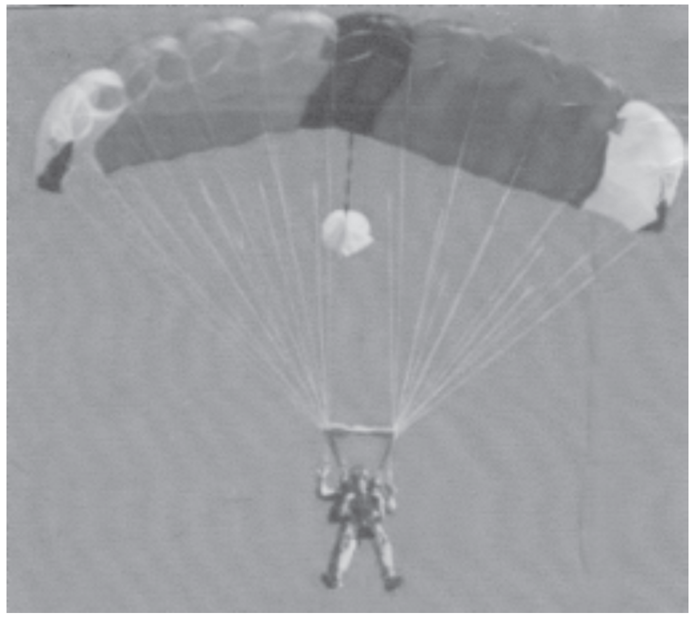
SELF ASSESSMENT TEST-1

1. (C) 2. (A) 3. (D) 4. (D) 5. (B)

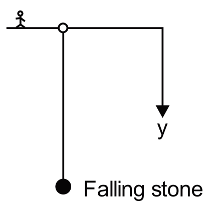
CHAPTER-9

DIFFERENTIAL EQUATIONS

Sky diving is a method of transiting from a high point in the atmosphere to the surface of the Earth with the aid of graity. This involves the control of speed during the descent using a parachute. Once the sky diver jumps from an airplane, the net force experienced by the diver can be calculated using DIFFERENTIAL EQUATIONS.



Another eg.



D.E. is

$$my'' = mg$$

$$\Rightarrow \quad y'' = g = \text{constant}$$

where y = distance travelled by the stone at any time t.

and g = acceleration due to gravity.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23

- Definition, order and degree
- General and particular solutions of a D.E.
- Solutions of D.E. using method of separation of variables.
- Solutions of homogeneous differential equations of first order and first degree.
- Solutions of linear differential equations of the type.

$\dfrac{dy}{dx} + py = q,$ where p and q are functions of x or constants.

$\dfrac{dx}{dy} + px = q,$ where p and q are functions of y or constants.

KEY POINTS :

- **DIFFERENTIAL EQUATION :** is an equation involving derivatives of the dependent variable w.r.t independent variables and the variables themselves.
- **ORDINARY DIFFERENTIAL EQUATION (ODE) :** A.D.E. involving derivatives of the dependent variable w.r.t only one independent variable is an ordinary D.E.

  In class XII ODE is referred to as D.E.
- **PARTIAL DIFFERENTIAL EQUATION (PDE) :** A.D.E involving derivatives w.r.t more than one independent variables is called a partial D.E.
- **ORDER of a DE :** is the order of the highest order derivative occurring in the D.E.
- **DEGREE of a D.E. :** is the highest power of the highest order derivative occurring in the D.E provided D.E is a polynomial equation in its derivatives.
- **SOLUTION OF THE D.E :** A relation between involved variables, which satisfy the given D.E is called its solution.

Two Types of Solution of DE

General Solution	**Particular Solution**
The solution which contains as many arbitrary constants as the order of the D.E.	The solution free from arbitrary constants

- **FORMATION OF A DIFFERENTIAL EQUATION :** We differentiate the function successively as many times as the arbitary constants in the given function and then eliminate the arbitiary constants from these equations.
- **ORDER of A D.E :** Is equal to the number of arbitrary constants in the general solution of a D.E.

Solution of a First Order First Degree D.E.

- Variable separable method
- Homogeneous D.E.
- Linear Differential equation

- **"VARIABLE SEPARABLE METHOD" :** is used to solve D.E. in which variables can be separated completely i.e, terms containing x should remain with dx and terms containing y should remain with dy.
- **"HOMOGENEOUS DIFFERENTIAL EQUATION :** D.E. of the form $\frac{dy}{dx} = F(x, y)$ where *F(x, y)* is a homogeneous function of degree 0

i.e. $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

or $F(\lambda x, \lambda y) = F(x, y)$ for some non-zero constant λ.

To solve this type put $y = vx$

To Solve homogenous D.E of the trype $\frac{dx}{dy} = G(x, y),$ we make substitution $x = vy$

- **LINEAR DIFFERENTIAL EQUATION :** A.D.E of the form $\frac{dy}{dx} + Py = Q$ where P and Q are constants or functions of x only is known as first order linear differential equation.

Its solution

$y.(IF) = \int Qx(I.F.)dx + C$ where

I. F = Integrating factor = $e^{\int P dx}$

Another form of Linear Differential Equation is $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are constants or functions of y only.

Its solution is given as

$x.(I.F) = \int Q_1 X(I.F.)\, dy + C,$ where $I.F. = e^{\int P_1 dy}$

Illustration:

Write the order and degree of the Differential Equation

$$\left[1 + (y')^2\right]^{3/2} = ky''$$

Solution: Squaring both the sides

$$\left[1 + (y')^2\right]^3 = k^2 (y'')^2$$

$\therefore$ Order of D.E. = 2

and Degree of D.E. = 2

Illustration:

Solve the differential equations

$$(1+e^{2x})dy + e^{x}(1+y^{2})dx = 0;\ y(0) = 1$$

Solution: $\dfrac{dy}{dx} = \dfrac{-e^{x}(1+y^{2})}{1+e^{2x}}$

Using Variables separables method,

$$\frac{dy}{1+y^2} = \frac{-e^x}{1+e^{2x}}\,dx$$

Integrating both sides we get

$$\int \frac{1}{1+y^2}\,dy = -\int \frac{e^x}{1+e^{2x}}\,dx$$

$\Rightarrow \tan^{-1} y = -\int \dfrac{dt}{1+t^2}$; On putting $e^x = t$

$= -\tan^{-1} t$

$\Rightarrow \tan^{-1} y = -\tan^{-1}(e^x) + C$

$\Rightarrow \tan^{-1} y + \tan^{-1}(e^x) = C$

At $x = 0$, $y = 1$ given

$\therefore\ \tan^{-1}(1) + \tan^{-1}(1) = C$

$\Rightarrow\ \dfrac{\pi}{4} \times 2 = C$

$\Rightarrow\ C = \dfrac{\pi}{2}$

$\therefore$ Particular solution of D.E. is given by $\tan^{-1} y + \tan^{-1}(e^x) = \dfrac{\pi}{2}$.

Illustration:

Solve $(x - y)\dfrac{dy}{dx} = x + 2y$

Solution: $\dfrac{dy}{dx} = \dfrac{x+2y}{x-y} = f(x, y)$

Now $f(\lambda x, \lambda y) = \dfrac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \dfrac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^{o} f(x, y)$

Clearly, *f* is homogeneous function in *x* and *y*.

So, given D.E. is **homogenous D.E.**

Now, Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + \frac{x\,dv}{dx}$$

$$\therefore \quad v + \frac{x\,dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow v + \frac{x\,dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \frac{(1-v)\,dv}{1+v+v^2} = \frac{dx}{x}$$

Integrating both sides we get

$$\Rightarrow -\frac{1}{2}\int \frac{2v-2+1-1}{1+v+v^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\int \frac{2v+1}{1+v+v^2}dv + \frac{3}{2}\int \frac{1}{1+v+v^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log|1+v+v^2| + \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dv = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right| + \left(\frac{3}{2}\right).\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \log|x| + C$$

$$\Rightarrow -\frac{1}{2}\log\left|x^2+xy+y^2\right| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = C$$

Illustration:

Find the particular solution of the differential equation

$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y\,(y \neq 0)$ given that $x = 0$ when $y = \pi/2$.

Solution: Clearly, it is a Linear D.E.

$\frac{dx}{dy} + Px = Q$ where

$P = \cot y$, $Q = 2y + y^2 \cot y$

I.F. = $e^{\int \text{Pdy}} = e^{\int \text{coty dy}} = e^{\int \log(\sin y)} = \sin y$

$\therefore$ solution of D.E. is given by

x. (I.F) = $\int$Q.I Fdy + C; C is arbitrary constant

$\Rightarrow$ $x.(\sin y) = \int (2y + y^2 \cot y)\ \sin y\ dy + C$

$= \int 2y \sin y\ dy + \int y^2 \cos y\ dy + C$

$= \int 2y \not{\sin y}\ dy + y^2 .\sin y - \int 2y \not{\sin y}\ dy + C$

$\Rightarrow$ $x \sin y = y^2 \sin y + C$

Now, $x = 0$, when $y = \frac{\pi}{2}$

So, $0 = \frac{\pi^2}{4} + C \Rightarrow C = -\frac{\pi^2}{4}$

$\therefore$ $x \sin y = y^2 \sin y - \frac{\pi^2}{4}$

or $\boxed{x = y^2 - \frac{\pi^2}{4} \text{ cosec } y}$

ONE MARK QUESTIONS

1. Write the order and degree of the following D.E.'s

(i) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \frac{d^2y}{dx^2}$ (ii) $\frac{d^5y}{dx^5}+log\left(\frac{dy}{dx}\right)=0$

(iii) $\sqrt{1+\frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}$ (iv) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{kd^2y}{dx^2}$

(v) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0$

2. (i) What will be the order of the D.E.

$$y = Ae^x + Be^{x+c}$$

(ii) What will be the order of the D.E. representing the family of circles with centre (o, a) and radius a.

(iii) Write the integrating factors of the following D Eqn.

(a) $\frac{dy}{dx}+y\cos x = \sin x$ (b) $\frac{xdy}{dx}+y\log x = x$

(c) $\frac{dy}{dx}+\frac{y}{x}=1$

(iv) State whether the following statements are True or False.

(a) Integrating factor of the D.E.

$(1-x^2)\frac{dy}{dx}-xy=1$ is $\sqrt{1-x^2}$

(b) Solution of D.E. xdy-ydx = 0 represents straight lines passing through origin.

(c) Number of arbitrary constants in the particular solution of a differential equation of order two is two.

TWO MARKS QUESTIONS

1. Write the general solution of the following D.Eqns.

   (i) $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$ (ii) $\frac{dy}{dx} = \frac{1 - \cos 2x}{1 + \cos 2y}$

   (iii) $(e^x + e^{-x})\, dy = (e^x - e^{-x})dx$

2. Given that $\frac{dy}{dx} = e^{-2y}$ and y = 0 when x = 5.

   Find the value of x when y = 3.

3. Name the curve for which the slope of the tangent at any point is equal to the ratio of the abbcissa to the ordinate of the point.

4. Solve $\frac{xdy}{dx} + y = e^x$.

THREE MARKS QUESTIONS

1. (i) Show that $y = e^{m \sin^{-1} x}$ is a solution of $(1 - x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} - m^2 y = 0$

   (ii) Show that $y = a \cos(\log x) + b \sin(\log x)$ is a solution of

   $$\frac{x^2 d^2 y}{dx^2} + \frac{xdy}{dx} + y = 0$$

   (iii) Verify that y = log $\left(x + \sqrt{x^2 + a^2}\right)$ satisfies the D.E.

   $$\left(a^2 + x^2\right) y'' + xy' = 0$$

2. Solve the following D Eqs.

   (i) $xdy - (y + 2x^2)dx = 0$

   (ii) $(1 + y^2)\tan^{-1} x\, dx + 2y(1 + x^2)dy = 0$

   (iii) $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

(iv) $\frac{dy}{dx} = 1 + x + y^2 + xy^2$, $y = 0$ when $x = 0$

(v) $xdy - ydx = \sqrt{x^2 + y^2}\,dx, y = 0$ when $x = 1$

3. Solve each of the following D Eqs.

(i) $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0,\ y(0) = 0$

(ii) $(x+1)\frac{dy}{dx} = 2e^{-y} - 1, y(0) = 0$

(iii) $e^x \tan y dx + (2 - e^x)\sec^2 y dy = 0,\ y\ (0) = \prod/4$

(iv) $(x^2 - y^2)dx + 2xydy = 0$

(v) $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}, y = 0$ when $x = 1$

4. Solve the following D.E.s.

(i) Find the particular solution of

$2y\ e^{x/y}\ dx + (y - 2xe^{x/y})\ dy = 0, x = 0$ if $y = 1$

(ii) $x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$

(iii) $\frac{dy}{dx} = \cos\ (x + y) + \sin\ (x + y)\ dx$

[**Hint :** Put $x + y = z$]

(iv) Show that the Differential Equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogenous and also solve it.

(v) $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}, |\ x\ | \neq 1$

FIVE MARKS QUESTIONS

Q. 1 Solve $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

Q. 2 Solve $(x\ dy - ydx)y \sin\left(\frac{y}{x}\right) = (ydx + xdy)x \cos\left(\frac{y}{x}\right)$

Q. 3 Find the particular solution of the D.E. $(x-y)\frac{dy}{dx} = x+2y$ given that $y=0$ when $x=1$.

Q. 4 Solve $dy = \cos x\,(2 - y\,\text{cosec}\,x)dx,$ given that $y=2$ when $x=\pi/2$

Q. 5 Find the particular solution of the D.E. $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$ given that $y=0$ when $x=1$

CASE STUDY QUESTIONS

Q. 1 Suppose a person named Devdutt saved Rs 10,000 from his earnings for his daughter's marriage.

So, he deposits this amount in a bank where principal amount increases continously at the rate of 10% per year. Now based on the following information answer the following questions.

(i) The following D.E. represents the above situation if P denotes the principal at time t.

(a) $\frac{dP}{dt} = \frac{10}{100}P$ (b) $\frac{dt}{dP} = \frac{10}{100}P$

(c) $\frac{dP}{dt} = \frac{100}{10}P$ (d) None of these

(ii) The order and degree of the D.E. obtained in (i) is

(a) order : 2 degree : 1

(b) order : 1, degree : 2

(c) order : 1, degree : not defined

(d) order : 1, degree : 1

(iii) The most suitable method for solving D.E. obtained in part (i) is

(a) Ordinary method

(b) Method for Homogeneous D.E.

(c) Variables separable method

(d) None of these

(iv) Solution of the D.E. obtained in part (i) is given by

(a) $C = Pe^{t/10}$ (b) $P = Ce^{t/10}$, C = arbitrary constant

(c) $P = Ce^{10/t}$ (d) $C = Pe^{10/t}$

(v) In how many years will ₹ 10,000 double itself.

(a) $10 \log_{10} 2$ (b) $2 \log e^{10}$

(c) $10 \log_e 2$ (d) $2 \log_{10} 10$

Self Assessment Test-1 Differential Equations

Q. 1 The general solution of the D.E.

$$\log\left(\frac{dy}{dx}\right) = ax + by \text{ is}$$

(a) $\frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + C$ (b) $e^{ax} - e^{-by} = C$

(c) $be^{ax} + ae^{by} = C$ (d) none of these

Q. 2 The general solution of the DE

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2 \text{ is}$$

(a) $\tan^{-1}\left(\frac{y}{x}\right) = \log x + c$ (b) $\tan^{-1}\left(\frac{x}{y}\right) = \log x + c$

(c) $\tan^{-1}\left(\frac{y}{x}\right) = \log y + c$ (d) none of these

Q. 3 The solution of the D.E.

$dy = (4 + y^2)\, dx$ is

(a) $y = 2 \tan (x + c)$ (b) $y = 2 \tan (2x + c)$

(c) $2y = \tan (2x + c)$ (d) $2y = 2 \tan (x + c)$

Q. 4 What is the degree of the D.E.

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^{-2}$$

(a) 1 (b) 3

(c) –2 (d) Degree doesn't exist

Q. 5 Solution of D.E. $xdy - ydx = 0$ represents:

(a) a rectangular hyperbola

(b) a parabola whose vertex is at the origin

(c) a straight line passing through the origin

(d) a circle whose centre is at the origin.

Self Assessment Test-2

Q. 1 $x\dfrac{dy}{dx} + 2y = x^2$ is

(a) $y = \dfrac{x^2 + c}{4x^2}$ (b) $y = \dfrac{x^2}{4} + c$

(c) $y = \dfrac{x^2 + c}{x^2}$ (d) $y = \dfrac{x^4 + c}{4x^2}$

Q. 2 The solution of the $\dfrac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is

(a) $y = e^{-x}(x - 1)$ (b) $y = x e^{x}$

(c) $y = xe^{-x} + 1$ (d) $y = x e^{-x}$

Q. 3 If $\dfrac{dy}{dx} = \dfrac{2^{x+y} - 2^x}{2^y}$, $y(0) = 1$, then $y(1)$ is equal to **[JEE mains 2021]**

(a) $\log_2(2 + e)$ (b) $\log_2(1 + e)$

(c) $\log_2(2e)$ (d) $\log_2(1 + e^2)$

Q. 4 If the solution curve of the D.E. $(2x - 10y^3)\,dy + y\,dx = 0$ passess through the points (0, 1) and $(2, \beta)$, then β is a root of the equation

(a) $y^5 - 2y - 2 = 0$ (b) $2y^5 - 2y - 1 = 0$

(c) $2y^5 - y^2 - 2 = 0$ (d) $y^5 - y^2 - 1 = 0$ **[JEE mains 2021]**

Q. 5 Consider a curve $y = f(x)$ passing through the point (–2, 2) and the slope of the tangent to the curve at any point $(x, f(x))$ is given by

$f(x) + xf^1(x) = x^2$, then,

(a) $x^3 + 2x\,f(x) - 12 = 0$ (b) $x^3 + xf(x) + 12 = 0$

(c) $x^3 - 3x\,f(x) - 4 = 0$ (d) $x^2 + 2xf(x) + 4 = 0$ **(HOTS)**

Answers

ONE MARK QUESTIONS

1.)i($0 \rightarrow 2, D \rightarrow 1$ (ii) $0 \rightarrow 5, D \rightarrow$ Not defined (iii) $0 \rightarrow 2, D \rightarrow 2$

   (iv) $0 \rightarrow 2, D \rightarrow 2$ (v) $0 \rightarrow 2, D \rightarrow$ Not defined

2. (i) Order = 1 (ii) Order = 1

(iii)(a) $e^{\sin x}$ (b) $e^{\frac{(\log x)^2}{2}}$ (c) x

(iv)(a True (b) True (c) False

TWO MARKS QUESTIONS

1. (i) $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + C$ (ii) $2(y - x) + \sin 2y + \sin 2x = c$

   (iii) $y = \log_e |e^x + e^{-x}| + C$

2. $\frac{e^6 + 9}{2}$ 3. Rectangular hyperbola 4. $\frac{d^2y}{dx^2} + y = 0$

5. $y.x = e^x + c$

THREE MARKS QUESTIONS

1. (i) $y = 2x^2 + cx$ (ii) $\frac{1}{2}(\tan^{-1}x)^2 + \log(1 + y^2) = c$

   (iii) $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$ (iv) $y = \tan\left(x + \frac{x^2}{2}\right)$

   (v) $y + \sqrt{x^2 + y^2} = x^2$

2. (i) $(1 + x^2)\, y = \frac{4x^3}{3}$ (ii) $(2 - e^y)(x + 1) = 1$

   (iii) $\tan y = 2 - e^x$ (iv) $x^2 + y^2 = cx$

   (v) $(1 + x^2)y = \tan^{-1}x - \pi/4$

3. (i) $e^{x/y} = \frac{-1}{2}\log|y| + 1$ (ii) $\sin(y/x) = \log|x| + c$

   (iii) $\log\left|1 + \tan\left(\frac{x + y}{2}\right)\right| = x + c$ (iv) $\frac{y}{x} - \log|y| = c$

   (v) $(x^2 - 1)y = \frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right| + c$

FIVE MARKS QUESTIONS

1. $y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x\log x}{3} - \frac{x}{9} + \frac{c}{x^2}$

2. $xy\cos\left(\frac{y}{x}\right) = c$

3. $\sqrt{3}\tan^{-1}\left(\frac{2y + x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2 + xy + y^2| = \frac{\sqrt{3}\pi}{6}$

4. $y\sin x = \frac{-1}{2}\cos(2x) + \frac{3}{2}$

5. $x = \frac{1}{2}e^{\tan^{-1}y} + \frac{1}{2}e^{-\tan^{-1}y}$

CASE STUDY QUESTIONS

1. (i) a (ii) d (iii) c
(iv) b (v) c

SELE ASSESSMENT TEST-1

1. (a) 2. (a) 3. (b)
4. (b) 5. (c)

SELE ASSESSMENT TEST-2

1. (d) 2. (d) 3. (b)
4. (d) 5. (c)

CHAPTER 10

VECTORS

Vectors are probably the most important tool to learn in all of physics and engineering. Vectors are using in daily life following are few of the example.

- Navigating by air and by boat is generally done using vectors.
- Planes are given a vector to travel, and they use their speed to determine how far they need to go before turning or landing. Flight plans are made using a series of vectors.
- Sports instructions are based on using vectors.

VECTORS

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

- Vectors and scalars
- Magnitude and direction of a vector
- Direction coiners and directions ratios of a vector.
- Types of vectors (equal, unit, zero, parallel and collinear vectors)
- Position vector of a point
- Negative of a vector
- Components of a vector
- Addition of vectors
- Multiplication of a vector by a scalar
- Position vector of a point dividing a line segment in a given ratio
- Definition, Geometrical interpretation, properties and application of scalar (dot) product of vectors
- Vector (cross) product of vectors.

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a vector. It is denoted by a directed line segment.

- Two or more vectors which are parallel to same line are called collinear vectors.

- Position vector of a point P(a, b, c) w.r.t. origin (0, 0, 0) is denoted by $\overrightarrow{OP}$ where $\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$.

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then

  $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and

  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z)^2}$

- Any vector $\vec{a}$ is called unit vector if $|\vec{a}| = 1$ It is denoted by $\hat{a}$

- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.

- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda\vec{a}$ is vector collinear with $\vec{a}$ and $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\vec{b}$ where λ is some scalar.

- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$ where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B, and C is any point which divides $\overrightarrow{AB}$ in ratio m:n internally then position vector $\vec{c}$ of point C is given as $\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$. If C divides $\overrightarrow{AB}$ in ratio m:n externally, then $\vec{c} = \frac{m\vec{b}-n\vec{a}}{m-n}$.

- The angles α, β *and* γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, y and z-axis are called angles and cosines of these angles are called direction cosines of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

  Also $l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$

- The numbers a, b, c proportional to l, m, n are called direction ratios.

- Scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}.\vec{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$, θ is the angle between $\vec{a}$ *and* $\vec{b}$. $(0 \le \theta \le \pi)$.

- Dot product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

- $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{o}$ or $\vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.

- $\vec{a}.\vec{a} = |\vec{a}|^2$, so $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot k = 1$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

- Projection of $\vec{a}$ on $\vec{b} = \left|\frac{\vec{a}.\ \vec{b}}{|\vec{b}|}\right|$ and

  Projection vector of $\vec{a}$ along $\vec{b} = \left(\frac{(\vec{a}.\ \vec{b})}{|\vec{b}|}\right)\hat{b}$.

- Cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$. were θ is the angle between $\vec{a}$ and $\vec{b}$.$(0 \leq \theta \leq \pi)$. And $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}\ .\vec{b}$ and $\hat{n}$ from a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$.

- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$.

- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$ and $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}, \hat{\imath} \times \hat{k} = -\hat{\jmath}$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$,then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\left(\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}\right)$.

- $|\vec{a} \times \vec{b}|$ is the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

- $\frac{1}{2}|\vec{a} \times \vec{b}|$ is the area of parallelogram where diagonals are $\vec{a}$ and $\vec{b}$.

- If $\vec{a}, \vec{b}$ and $\vec{c}$ form a triangle, then area of the triangle

- $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}|$.

Illustration:

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} + 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$ Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 27$

Solution:

$\because$ $\vec{d}$ is perpendicular to $\vec{a}$ and $\vec{b}$ *both*

$$\text{Let } \vec{d} = \lambda(\vec{a} \times \vec{b}) = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$\vec{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k})$$

But $\vec{c} \cdot \vec{d} = 27$

$\therefore$ $(2\hat{i} - \hat{j} + 4\hat{k}) . \lambda(32\hat{i} - \hat{j} + 14\hat{k}) = 27$

$\Rightarrow$ $\lambda(64 + 1 - 56) = 27$

$\Rightarrow$ $\lambda = 3$

and $\vec{d} = 3(32\hat{i} - \hat{j} - 14\hat{k}) = 96\hat{i} - 3\hat{j} + 42\hat{k}$

Illustration:

Vectors $\vec{a}$, $\vec{b}$ and $\vec{b}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 5, |\vec{b}| = 7$ and $|\vec{c}| = 3$.

Find the angle between $\vec{a}$ and $\vec{c}$

Solution:

Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\vec{a} + \vec{c} = -\vec{b}$$

$$(\vec{a} + \vec{c}) \cdot (\vec{a} + \vec{c}) = (-\vec{b}) \cdot - (-\vec{b})$$

$\Rightarrow (\vec{a})^2 + \vec{a} \cdot \vec{c} + \vec{c} \cdot \vec{a} + (\vec{c})^2 = |\vec{b}|^2$ $\quad (\because \vec{a} \cdot \vec{a} = |\vec{a}|^2)$

$\Rightarrow$ $2\vec{a} \cdot \vec{c} = |\vec{b}|^2 - |\vec{a}|^2 - |\vec{c}|^2$

$\Rightarrow$ $2|\vec{a}||\vec{c}|\cos\theta = |\vec{b}|^2 - |\vec{a}|^2 - |\vec{c}|^2$

Where 'θ' be the angle between $\vec{a}$ and $\vec{c}$

$\Rightarrow 2 \times 5 \times 3 \cos\theta = 49 - 25 - 9$

$\Rightarrow \cos\theta = \dfrac{15}{30}$

$\Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\Pi}{3}$

Illustration:

Let $\vec{a}$ and $\vec{b}$ are two unit vectors and 'θ' is the angle between them, then find 'θ' if $\vec{a}+\vec{b}$ is unit vector.

Solution:

Here $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

$\because |\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 1$ $\qquad (\because \vec{a}\cdot\vec{a} = |\vec{a}|^2)$

$\Rightarrow (\vec{a})^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + (\vec{b})^2 = 1$

$\Rightarrow 2|\vec{a}||\vec{b}|\cos\theta = -1$

$\Rightarrow \cos\theta = -\dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{2\pi}{3}.$

ONE MARK QUESTIONS

1. If $\overrightarrow{AB} = 3\hat{i} + 2\hat{j} - \hat{k}$ and the coordinate of A are (4,1,1), then find the coordinates of B·

2. Let $\vec{a} = -2\hat{i}+\hat{j}$, $\vec{b} = \hat{i}+2\hat{j}$ and $\vec{c} = 4\hat{i}+3\hat{j}$. Find the values of x and y such that $\vec{c} = x\vec{a} + y\vec{b}$.

3. Find a unit vector in the direction of the resultant of the vectors $\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}+2\hat{j}-2\hat{k}$.

4. Find a vector of magnitude of 5 units parallel to the resultant of vector $\vec{a} = 2\hat{\imath}+3\hat{\jmath}+\hat{k}$ and $\vec{b} = (\hat{\imath} - 2\hat{\jmath} - \hat{k})$

5. For what value of λ the vector $\vec{a}$ *and* $\vec{b}$ perpendicular to each other? Where $\vec{a} = \lambda\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{b} = 5\hat{\imath}-9\hat{\jmath}+2\hat{k}$

6. Write the value of p for which $\vec{a} = 3\hat{\imath}+2\hat{\jmath}+9\hat{k}$ and $\vec{b} = \hat{\imath}+p\hat{\jmath}+3\hat{k}$ are parallel vectors.

7. For any two vectors $\vec{a}$ *and* $\vec{b}$ write when $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ holds.

8. Find the value of p if $\left(2\hat{\imath}+6\hat{\jmath}+27\hat{k}\right)\times\left(\hat{\imath}+3\hat{\jmath}+p\hat{k}\right)=\vec{0}$

9. Evaluate: $\hat{\imath}.\left(\hat{\jmath}\times\hat{k}\right)+\left(\hat{\imath}\times\hat{k}\right).\hat{\jmath}$

10. If $\vec{a}=2\hat{\imath}-3\hat{\jmath},\ \vec{b}=\hat{\imath}+\hat{\jmath}-\hat{k}, \vec{c}=\ 3\hat{\imath}-\hat{k}$, find $\vec{a}\cdot\vec{b}\times\vec{c}$

11. If $\vec{a}=5\hat{\imath}-4\hat{\jmath}+\hat{k},\ \vec{b}=-4\hat{\imath}+3\hat{\jmath}-2\hat{k}$ and $\vec{c}=\hat{\imath}-2\hat{\jmath}-2\hat{k}$, then evaluate $\vec{c}.(\vec{a}\times\vec{b})$

12. If $\vec{a}=p\hat{i}+3\hat{j}$ and $\vec{b}=4\hat{i}+p\hat{j}$, Find the values of p so that $\vec{a}$ and $\vec{b}$ may be collinear

13. Find a vector of magnitude 6 which is perpendicular to both the vectors $2\hat{\imath}-\hat{\jmath}+2\hat{k}$ and $4\hat{\imath}-\hat{\jmath}+3\hat{k}$.

14. If $\vec{a}\,.\,\vec{b}=0$, then what can you say about $\vec{a}\ and\ \vec{b}$?

15. If $\vec{a}\ and\ \vec{b}$ are two vectors such that $\left|\vec{a}\times\vec{b}\right|=\vec{a}.\vec{b}$, then what is the angle between $\vec{a}\ and\ \vec{b}$?

16. Find the area of a parallelogram having diagonals $3\hat{\imath}+\hat{\jmath}-2\hat{k}$ and$\hat{\imath}-$ $\hat{\imath}-3\hat{\jmath}+4\hat{k}$.

17. If $\hat{\imath},\hat{\jmath}\ and\ \hat{k}$ are three mutually perpendicular vectors, then find the value of $\hat{\jmath}.(\hat{k}\times\hat{\imath})$.

18. P and Q are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of a point R which divides the segment PQ in the ratio 2:1 externally.

19. Find λ when scalar projection of $\vec{a}=\lambda\hat{\imath}+\hat{\jmath}+4\hat{k}$ on $\vec{b}=2\hat{\imath}+6\hat{\jmath}+3\hat{k}$ is 4 units.

20. Find "a" so that the vectors $\vec{p}=3\hat{\imath}-2\hat{\jmath}$ and $\vec{q}=2\hat{\imath}+a\hat{\jmath}$ be orthogonal.

21. If $\vec{a}=\hat{\imath}-\hat{\jmath}+\hat{k},\ \vec{b}=2\hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{c}=\lambda\hat{\imath}-\hat{\jmath}+\lambda\hat{k}$ such that $\vec{a}\cdot(\vec{b}\times\vec{c})=0$ find the value of λ.

22. What is the point of trisection of PQ nearer to P if positions of P and Q are $3\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ and $9\hat{\imath} + 8\hat{\jmath} - 10\hat{k}$ respectively?

23. What is the angle between $\vec{a}$ and $\vec{b}$, if $\vec{a}.\vec{b} = 3$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$.

24. Represent graphicaley a displacement of 50 km, 60° south of west.

25. If the vectors $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{b} = \hat{j}$ and $\hat{c}$ are such that $\vec{a}$, $\vec{c}$ and $\vec{b}$ from a right handed system, than find $\vec{c}$.

TWO MARK QUESTIONS

1. A vector $\vec{r}$ is inclined to x – axis at 45° and y-axis at 60° if | $\vec{r}$ | = 8 units. find $\vec{r}$.

2. if $|\vec{a}+\vec{b}|$ = 60, $|\vec{a}-\vec{b}|$ = 40 and $\vec{b}$ = 46 find $|\vec{a}|$

3. Write the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where

   $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

4. If the points (–1, –1, 2), (2, m, 5) and (3, 11, 6) are collinear, find the value of *m*.

5. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ write value of the following.
   $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

6. If $(\vec{a} \times \vec{b})^2 + (\vec{a}.\vec{b})^2$ = 144 and $|\vec{a}|$ = 4. Find the value of $|\vec{b}|$.

7. If for any two vectors $\vec{a}$ and $\vec{b}$,

   $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = \lambda\left[(\vec{a})^2 + (\vec{b})^2\right]$ then write the value of λ .

8. if $\vec{a}, \vec{b}$ are two vectors such that $|(\vec{a}+\vec{b})| = |\vec{a}|$ then prove that $2\vec{a} + \vec{b}$ is perpendicular to $\vec{b}$.

9. Show that vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angle triangle.

10. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}$ = 0 and $|\vec{a}|$ = 5, $|\vec{b}|$ = 12, $|\vec{c}|$ = 13, then find $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

11. The two vectors $\hat{i} + \hat{j}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC respectively of Δ ABC, find the length of median through A.

12. If position vectors of the points *A*, *B* and *C* are $\vec{a}$, b and $4a - 3b$ respectively, then find vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

13. If position vectors of three points *A*, *B* and *C* are $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$ and $7\vec{a} - c$ respectively. Then prove that *A*, *B* and *C* are collinear.

14. If the vector $\hat{i} + p\hat{j} + 3\hat{k}$ is rotated through an angle θ and is doubled in magnitude, then it becomes $4\hat{i} + (4p - 2)\hat{j} + 2\hat{k}$. Find the value of *p*.

15. If $\overrightarrow{AB} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ and $\overrightarrow{AC} = 3\hat{i} + 4\hat{k}$ are sides of the triangle *ABC*. Find the length of median through *A*.

16. Find scalar projection of the vector $7\hat{i} + \hat{j} + 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$. Also find vector porojection

17. Let $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ one mutually perpendicular and $|\vec{a}| = |\vec{b}|$. Find *x* and *y*.

18. If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}$ and $\vec{b}$ so that $\vec{a} - \sqrt{2}\,\vec{b}$ is a unit vector.

19. If $\vec{a} = 2\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{z} + 3\hat{j} - 5\hat{k}$. Find the angle between $\vec{a}$ and $\vec{a} \times \vec{b}$.

20. Using vectors, prove that angle in a semi circle is 90°.

THREE MARKS QUESTIONS

1. The points A,B and C with position vectors $3\hat{\imath} - y\hat{\jmath} + 2\hat{k},\ 5\hat{\imath} - \hat{\jmath} + \hat{k}\ and\ 3x\hat{\imath} + 3\hat{\jmath} - \hat{k}$ are collinear. Find the values of x and y and also the ratio in which the point B divides AC.

2. If sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

3. Let $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k},\ \vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}\ and\ \vec{b}$ and satisfying $\vec{d}.\vec{c} = 21$

4. If $\hat{a}\ and\ \hat{b}$ are unit vectors inclined at an angleθ then proved that

   (i) $\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} + \hat{b}\right|$

   (ii) $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

   (iii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} - \hat{b}}\right|$

5. If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of equal magnitude. Prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a},\vec{b}$ and $\vec{c}$. Also find angles.

6. For any vector $\vec{a}$ prove that $|\vec{a} \times \hat{\imath}|^2 + |\vec{a} \times \hat{\jmath}|^2 + \left|\vec{a} \times \hat{k}\right|^2 = 2|\vec{a}|^2$

7. Show that $\left(\vec{a} \times \vec{b}\right)^2 = |\vec{a}|^2\left|\vec{b}\right|^2 - \left(\vec{a}.\vec{b}\right)^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

8. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are the position vectors of vertices A,B,C of a Δ ABC, show that the area of triangle ABC is $\frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$. Deduce the condition for points $\vec{a}, \vec{b}\ and\ \vec{c}$ to be collinear.

9. Let $\vec{a}, \vec{b}\ and\ \vec{c}$ be unit vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$ and the angle between b and c is $\pi/6$, prove that $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

10. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

11. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k},\ \vec{c} = \hat{\jmath} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$.

12. Find the altitude of a parallelepiped determined by the vectors $\vec{a}, \vec{b}\ and\ \vec{c}$ if the base is taken as parallelogram determined by $\vec{a}\ and\ \vec{b}$ and if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k},\ \vec{b} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

13. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$, such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

14. If $|\vec{a}| = 3,\ |\vec{b}| = 4\ and\ |\vec{c}| = 5$ such that each is perpendicular to sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$

15. Decompose the vector $6\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$ in two vectors which are parallel and perpendicular to the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively.

16. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c},\ \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, a$\neq 0$, then show that $\vec{b} = \vec{c}$.

17. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are three non zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$. Prove that $\vec{a}, \vec{b}\ and\ \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$

18. Simplify $(\vec{a}-\vec{b})\cdot\{(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})\}$

19. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value of $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j}) + xy$

20. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

21. The magnitude of the vector product of the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ with a unit vector along the sum of the vector $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{\imath}+2\hat{\jmath}+3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

22. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, prove that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

23. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = 3\hat{i}-\hat{j}+2\hat{k}$.

24. Prove that the angle betwen two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

25. If $\vec{\alpha} = 3\hat{i}-\hat{j}$ and $\vec{\beta} = 2\hat{i}+\hat{j}+3\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

26. Find a unit vector perpendicular to plane ABC, when position vectors of A,B,C are $3\hat{\imath}-\hat{\jmath}+2\hat{k}$, $\hat{\imath}-\hat{\jmath}-3\hat{k}$ and $4\hat{\imath}-3\hat{\jmath}+\hat{k}$ respectively.

27. Find a unit vector in XY plane which makes an angle 45° with the vector $\hat{\imath} + \hat{\jmath}$ at angle of 60° with the vector $3\hat{\imath} - 4\hat{\jmath}$.

28. Suppose $\vec{a} = \lambda\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$, $\vec{b} = \lambda\hat{\imath} + \hat{\jmath} + 2\lambda\hat{k}$. If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90°, then prove that λ satisfies the inequality $-7 < \lambda < 1$.

29. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$ then find the value of $(2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b})$.

30. Let $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$. Find a vector $\vec{d}$ such that $\vec{a}.\vec{d} = 0$, $\vec{b}.\vec{d} = 2$ and $\vec{c}.\vec{d} = 4$.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECTION CHOOSE THE CORRECT OPTION.

1. A unit vector perpendicular to both $\hat{i} + \hat{j}$ *and* $\hat{j} + \hat{k}$ is

   (A) $\hat{i} + \hat{j} + \hat{k}$ (B) $\hat{i} - \hat{j} + \hat{k}$

   (C) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$ (D) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

2. If $|\vec{a} \cdot \vec{b}| = 2$, $|\vec{a} \times \vec{b}| = 4$, then the value of $|\vec{a}|^2 \, |\vec{b}|^2$ is

   (A) 2 (B) 6

   (C) 8 (D) 20

3. The projection of vector $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ on vector $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$ is

   (A) $\frac{9}{19}$ (B) $\frac{9}{\sqrt{19}}$

   (C) $\frac{9}{\sqrt{6}}$ (D) $\frac{19}{9}$

4. If $\vec{a}$ is any vector, then the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

(A) $|\vec{a}|^2$ (B) $2|\vec{a}|^2$

(C) $3|\vec{a}|^2$ (D) $4|\vec{a}|^2$

5. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}, |\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{3}$

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECTION CHOOSE THE CORRECT OPTION.

1. If $\vec{a} \cdot \vec{b}$ and $\vec{a} + \vec{b}$ are unit vectors. Then the value of $|\vec{a} - \vec{b}|$ is

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $\sqrt{3}$

2. If $\vec{a}$ and $\vec{b}$ are two vectors *such that* $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$, then the value of $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$ is

(A) 0 (B) 41 (C) 29 (D) 7

3. If $\vec{c} \cdot (\hat{i} + \hat{j}) = 2, \vec{c} \cdot (\hat{i} - \hat{j}) = 3$ and $\vec{c} \cdot \hat{k} = 0$, then the vector $\vec{c}$ is

(A) $\frac{1}{2}(5\hat{i} + \hat{j})$ (B) $\frac{1}{2}(5\hat{i} - \hat{j})$

(C) $\frac{1}{2}(\hat{i} - 5\hat{j})$ (D) $\frac{1}{2}(\hat{i} + 5\hat{j})$

4. If the project of $3\hat{i} + \lambda\hat{j} + \hat{k}$ *on* $\hat{i} + \hat{j}$ is $\sqrt{2}$ units, then the value λ is

(A) 1 (B) –1 (C) 0 (D) 2

5. If $|\vec{a}| = 2, |\vec{b}| = 7$ *and* $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} - 6\hat{k}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

Answers

ONE MARK QUESTIONS

1. (7, 3, 0)
2. $x = -1,\ y = 2$
3. $\frac{1}{\sqrt{21}}(4\hat{i} + 2\hat{j} - \hat{k})$
4. $\sqrt{\frac{5}{2}}(3\hat{i} + \hat{j})$
5. $\lambda = \frac{16}{5}$
6. $2/3$
7. $\vec{a}$ *and* $\vec{b}$ *are perpendicular*
8. $\frac{27}{2}$
9. 0
10. 4
11. –5
12. p = $\pm 2\sqrt{3}$
13. $-2\hat{i} + 4\hat{j} + 4\hat{k}$
14. Either $\vec{a}$ = 0 or $\vec{b}$ = 0 or $\vec{a} \perp \vec{b}$
15. 45°
16. $5\sqrt{3}$ sq. Units
17. 1

18. $-\vec{a} + 4\vec{b}$

19. $\lambda = 5$

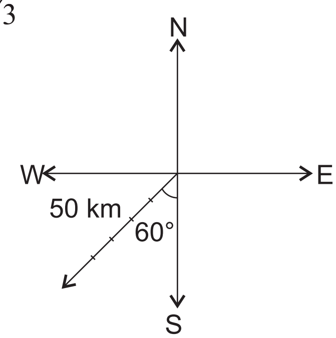
20. a = 3

21. $\lambda = 1$

22. $\left(5, \frac{14}{3}, -6\right)$

23. $\pi/3$

24.



25. $z\hat{j} - x\hat{k}$

TWO MARK QUESTIONS

1. $4(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$
2. 22
3. 2
4. $m = 8$
5. 0
6. 3
7. $\lambda = 2$
8. —
9. —
10. –169
11. $2\sqrt{2}$

12. $\overrightarrow{AC} = 3(\vec{a} - \vec{b})$, $\overrightarrow{BC} = 4(\vec{a} - \vec{b})$

14. $p = \frac{2}{3}, 2$

15. $\sqrt{33}$

16. $\frac{32}{7}$, $\frac{32}{49}$, $(2\hat{i} + 6\hat{j} + 3\hat{k})$

17. $x = -\frac{31}{7}$ $y = \frac{41}{12}$

18. $\frac{\pi}{4}$

19. $\frac{\pi}{2}$

THREE MARKS QUESTIONS

1. $x = 3,\ y = 3,\ 1:2$

3. $\vec{d} = 7\hat{\imath} - 7\hat{\jmath} - 7\hat{k}$

5. $\cos^{-1}\frac{1}{\sqrt{3}}$

11. $\vec{b} = \frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$

12. $\frac{4}{\sqrt{38}}$ units

13. $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

14. $5\sqrt{2}$

15. $(-\hat{\imath} - \hat{\jmath} - \hat{k}) + (7\hat{\imath} - 2\hat{\jmath} - 5\hat{k})$

18. 0

19. 0

20. 60°

21. $\lambda = 1$

23. $\hat{i} - 11\hat{j} - 7\hat{k}$

25. $\vec{\beta} = \left(\frac{3}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\right) + \left(\frac{1}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} + 3\hat{k}\right)$

26. $\frac{-1}{\sqrt{165}}(10\hat{\imath} + 7\hat{\jmath} - 4\hat{k})$

27. $\frac{13}{\sqrt{170}}\hat{i} + \frac{1}{\sqrt{170}}\hat{j}$

29. $-\frac{11}{2}$

30. $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

SELF ASSESSMENT-1

1. (C) 2. (D)
3. (D) 4. (B)
5. (B)

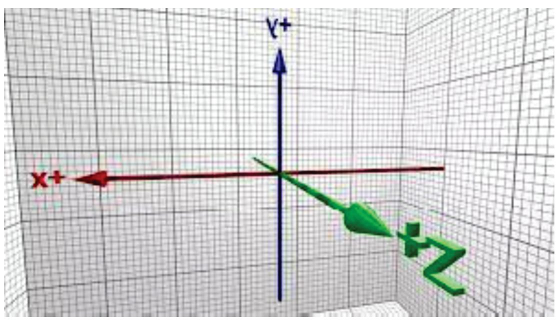
SELF ASSESSMENT-2

1. (D) 2. (A)
3. (B) 4. (B)
5. (A)

CHAPTER 11

THREE-DIMENSIONAL GEOMETRY

In the real world, everything you see is in a three-dimensional shape, it has length, breadth, and height. Just simply look around and observe. Even a thin sheet of paper has some thankless.



Applications of geometry in the real world include the computer-aided design (CAD) for construction blueprints, the design of assembly systems in manufacturing such as automobiles, nanotechnology, computer graphics, visual graphs, video game programming, and virtual reality creation.

The next time you play a mobile game, thank three-dimension geometry for the realistic look to the landscape and the characters that inhabit the game's virtual world.

THREE DIMENSIONAL GEOMETRY

Topics to be covered as per C.B.S.E. revised syllabus (2022-23)

- Direction cosines and direction ratios of a line joining two points.
- Certersion equation and vector equation of a line.
- Shortest distance between two lines.
- Angle between two lines.

POINTS TO REMEMBER

- **Distance Formula:** Distance (d) between two points(x_1 , y_1 , z_1)and(x_2, y_2, z_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** line segment AB is divided by P (x, y, z) in ratio m:n

(a) Internally	**(b) Externally**
$\left(\frac{m x_2 + n x_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

- **Direction ratio** of a line through $(x_1, y_1, z_1)$and (x_2, y_2, z_2) are$x_2 - x_1, y_2 - y_1, z_2 - z_1$

- **Direction cosines** of a line having direction ratios as a, b, c are:

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of line in space:**

Vector form	**Cartesian form**
(i) Passing through point $\vec{a}$ and parallel to vector $\vec{b}$; $\vec{r} = \vec{a} + \lambda\vec{b}$	(i) Passing through point (x_1, y_1, z_1) and having direction ratios a, b, c;

	$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$
(ii)Passing through two points $\vec{a}$and$\vec{b}$; $\vec{r}=\vec{a}+\lambda\,(\vec{b}-\vec{a})$	(ii) Passing through two points $(x_1, y_1 z_1)$ and $(x_2, y_2 z_2)$; $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

- **Angle between two lines:**

Vector form	Cartesian form
(i) For lines $\vec{r}=\overrightarrow{a_1}+\lambda\,\overrightarrow{b_1}$ and $\vec{r}=\overrightarrow{a_2}+\mu\,\overrightarrow{b_2}$, $\cos\theta=\frac{\lvert\vec{b}_1\,.\,\vec{b}_2\rvert}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}$ where 'θ' is the angle between two lines.	(ii) For lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ $\cos\theta$ $=\frac{\lvert a_1a_2\;+\;b_1b_2\;+\;c_1c_2\rvert}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
(iii) Lines are perpendicular if $\vec{b}_1\,.\,\vec{b}_2=0$	(ii) Lines are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$
(iv) Lines are parallel if $\vec{b}_1=k\vec{b}_2$; $k\neq 0$	(i) Lines are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

- **Shortest distance between two skew lines**

The shortest distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $$d = \left\lvert\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{\lvert\vec{b}_1 \times \vec{b}_2\rvert}\right\rvert$$ If d = 0, lines are intersecting	The shortest distance between $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$$ Where $D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$

- **Shortest distance between two parallel lines**

Let $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ are parallel lines then shortest distance between those lines

$$d = \left\lvert\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{(\vec{b})}\right\rvert \text{ units}$$

If d = 0, then lines coincident.

Illustration 1:
Are the following lines interesting?

$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

and $\vec{r} = 5\hat{i} + 2\hat{j} - \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

If yes, find point of intersection.

Solution:

We can write the equations is Certesion form

$$\frac{x-3}{1}=\frac{y-2}{2}=\frac{z+4}{2}=\lambda \quad ...(i)$$

and $$\frac{x-5}{3}=\frac{y+2}{2}=\frac{z}{6}=\mu \quad ...(ii)$$

Any point on line (i) P(λ + 3, 2λ +2, 2λ – 4)

Any point on line (ii) Q (3μ + 5, 2μ – 2, 6μ)

Comparing *x*, *y* and *z* coordinate respectively

$\lambda + 3 = 3\mu + 5,\ 2\lambda + 2 = 2\mu - 2,\ 2\lambda - 4 = 6\mu$

or $\lambda - 3\mu = 2,\ 2\lambda - 2\mu = -4,\ 2\lambda - 6\mu = 4$

or $\lambda - 3\mu = 2,\ \lambda - \mu = -2,\ \lambda - 3\mu = 2$

Solving first two, we get $\lambda = -4$, $\mu = -2$

$\because$ $\lambda = -4$, $\mu = -2$, Satisfies $\lambda - 3\mu = 2$

$\therefore$ lines are intersecting

and point of interesting (–1, –6, –12)

Or

Using distance formula

If $(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) = 0$

then lines are intersecting

Illustration 2:

Find the foot of perpendicular from the point P(1, 2, 3) to the line $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$.

Also find the length of the perpendicular and image of P in the given lines.

Solution: We have

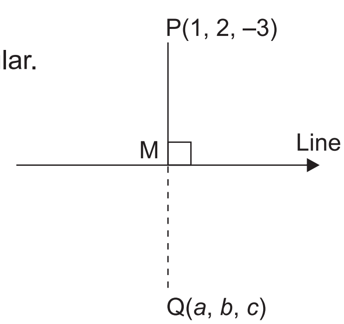
$$\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}=\lambda \text{ (say)}$$

$\therefore$ $x = 2\lambda - 1$, $y = -2\lambda + 3$, $z = -\lambda$

Let M($2\lambda - 1$, $-2\lambda + 3$, $-\lambda$) be the foot of perpendicular.

DR's of PM are <$2\lambda - 1 - 1$, $-2\lambda + 3 - 2$, $-\lambda + 3$>

or < $2\lambda - 2$, $-2\lambda + 1$, $-\lambda + 3$>



$\because$ PM is perpendicular to the line

$\therefore$ $2(2\lambda - 2) - 2(-2\lambda + 1) - 1(-\lambda + 3) = 0$

$4\lambda - 4 + 4\lambda - 2 + \lambda - 3 = 0$

$9\lambda - 9 = 0$

$\Rightarrow \lambda = 1$

$\therefore$ Foot of the perpendicular m = (1, 1 – 1)

and PM = $\sqrt{(1-1)^2+(2-1)^2+(-3+1)^2} = \sqrt{0+1+4} = \sqrt{5}$

Let Q(*a*, *b*, *c*) be the image of P

As m be the mid point of PQ. (As line is plane mirror)

$\therefore \quad \frac{a+1}{2} = 1 \quad \Rightarrow \quad a = 1$

$\frac{b+2}{2} = 1 \quad \Rightarrow \quad b = 0$

$\frac{c-3}{2} = -1 \quad \Rightarrow \quad c = 1$

$\therefore$ image of P is (1, 0, 1)

ONE MARK QUESTIONS

1. What is the distance of point (a, b, c) from x-axis?

2. What is the angle between the lines $2x = 3y = -z$ and $6x =- y = -4z$?

3. Write the equation of a line passing through (2, -3, 5) and parallel to line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$.

4. Write the equation of a line through (1, 2, 3) and parallel to $\vec{r}.(\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 5$.

5. What is the value of λ for which the lines $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$ are perpendicular to each other?

6. Write line $\vec{r} = (\hat{\imath} - \hat{\jmath}) + \lambda\left(2\hat{\jmath} - \hat{k}\right)$ into Cartesian form.

7. If the direction ratios of a line are 1, -2, 2 then what are the direction cosines of the line?

8. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.

9. What is the perpendicular distance of plane $2x - y + 3z = 10$ from origin?

10. If O is origin OP = 3 with direction ratios proportional ratio –1, 2, –2 ten what are the coordinate of P?

11. Write the line $2x = 3y = 4z$ in vector form.

12. Write direction ratios and direction cosines of z-axis.

13. Write direction ratios and direction cosines of the line $\frac{x+1}{3} = \frac{y-1}{-1}, z + 1 = 0.$

14. The cartesian equations of a line are x = ay + b, z = cy + d. Find direction ratios of the line, also write its equation in vector form.

TWO MARK QUESTIONS

1. Find the equation of a line passing though (2, 0, 5) and which is parallel to line $6x - 2 = 3y + 1 = 2z - 2$

2. The equation of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line

3. If a line makes angle α, β, γ with Co-ordinate axis then what is the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$

4. Find the equation of a line passing through the point (2, 0, 1) and parallel to the line whose equation is $\vec{r} = (2\lambda + 3)\hat{i} + (7\lambda - 1)\hat{j} + (-3\lambda + 2)\hat{k}$

5. Find the condition that the lines x = ay + b, z = cy + d and x = a′y + b′, z = c′y + d′ may be perpendicular to each other.

6. Show that the lines x = –y = 2z and x + 2 = 2y – 1 = –z + 1 are perpendicular to each other.

7. Find the equation of the line through (2, 1, 3) and parallel to the line $\frac{2x-1}{2}=\frac{4-y}{7}=\frac{z+1}{2}$ in Certesion and vector form)

8. Find the Certesian and vector equation of the line through the points (2, –3, 1) and (3, –4, –5)

9. For what value of λ and μ the line joining the points (7, λ, 2), (μ, –2, 5) is parallel to the line joining the points (2, –3, 5), (–6, –15, 11)?

10. If the points (–1, 3, 2), (–4, 2, –2) and (5, 5, λ) are Collinear, find the value of λ.

THREE MARKS QUESTIONS

1. Find vector and Cartesian equation of a line passing through a point with position vector $2\hat{\imath}-\hat{\jmath}+\hat{k}$ and which is parallel to the line joining the points with position vectors $-\hat{\imath}+4\hat{\jmath}+\hat{k}$ and $\hat{\imath}+2\hat{\jmath}+2\hat{k}$.

2. Find image (reflection) of the point (7, 4, –3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

3. Show that the lines line $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and line $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect each other. Find the point of intersection.

4. Find the shortest distance between the lines:

$$\overline{r} = \hat{\imath}+2\hat{\jmath}+3\hat{k}+\mu\left(2\hat{\imath}+3\hat{\jmath}+4\hat{k}\right) \text{ and}$$

$$\overline{r}=\left(2\hat{\imath}+4\hat{\jmath}+5\hat{k}\right)+\lambda\left(3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right).$$

5. Find shortest distance between the lines:

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{5-y}{2} = \frac{z-7}{1}$

6. Find the shortest distance between the lines:

$\overline{r} = (1-\lambda)\hat{\imath} + (\lambda - 2)\hat{\jmath} + (3-2\lambda)\hat{k}$

$$\overline{r} = (\mu+1)\hat{\imath} + (2\mu - 1)\hat{\jmath} - (2\mu+1)\hat{k}$$

7. Find the foot of perpendicular from the point $2\hat{\imath} - \hat{\jmath} + 5\hat{k}$ on the line $\overline{r} = (11\hat{\imath} - 2\hat{\jmath} - 8\hat{k}) + \lambda(10\hat{\imath} - 4\hat{\jmath} - 11\hat{k})$. Also find the length of the perpendicular.

8. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonal of a cube. Prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3}$

9. Find the length and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

10. Show that $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{2}, z = 2.$ do not intersect each other.

11. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then find the value of *k*.

12. Find the equation of the line which intersects the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$ and passes through the point (1, 1, 1).

13. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angle of $\pi/3$.

14. Find the foot of perpendicular drawn from the point (2, –1, 5) to the line

$$\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$$

Also find the length of the perpendicular. Hence find the image of the point (2, –1, 5) in the given line.

15. Find the image of the point P(2, –1, 11) in the line

$$\vec{r} = (2\hat{i} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$

16. Find the point(s) on the line through the point P(3, 5, 9) and Q(1, 2, 3) at a distance 14 units from the mid-point of segment PQ.

17. Find the shortest distance between the following pair of lines

$$\frac{x-1}{2} = \frac{y+1}{3} = z \text{ and } \frac{x+1}{5} = \frac{y-2}{1};\ z = 2$$

Hence write whether the lines are intersecting or not.

18. Find the foot of perpendicular from the point (1, 2, 3) to the line

$$\vec{r} = (6\hat{i} + 7\hat{j} + 7\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

Also find the equation of the perpendicular and length of perpendicular.

19. Find the equation of the line passing through (–1, 3, –2) and perpendicular to the lines $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z-5}{3}$ and $\frac{x-2}{-3} = \frac{y}{2} = \frac{z+1}{5}$

20. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{3-z}{-2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3)

21. The points P(4, 5, 10), Q(2, 3, 4) and R(1, 2, –1) are three vertices of a parallelogram PQRS. Find the vector equations of the sides PQ and QR and also find the coordinates of point R.

22. Find the equation of perpendicular from the point (3, –1, 11) to the line

$$\frac{x}{2} = \frac{2y-4}{6} = \frac{3-z}{-4}.$$

Also find the foot of the perpendicular and the length of the perpendicular.

23. Show that the lines $\frac{1-x}{-2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{1+y}{2} = z$ are intersecting. Also find the point of intersection.

24. For what value of 'λ', the following are Skew lines?

$$\frac{x-4}{5}=\frac{1+y}{2}=z,\ \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-\lambda}{4}$$

25. Find the vector equation of the line passing through (2, 1, –1) and parallel to the line $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$. Also find the distance between these two lines.

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. The foot of perpendicular drawn from the point (2, –1, 5) to the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$ is

   (a) (2, 1, 3) (b) (3, 1, 2)

   (c) (1, 2, 3) (d) (3, 2, 1)

2. The shortest distance between the lines $\vec{r}=(6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$ and $\vec{r}=(-4\hat{i}-4\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$ is

   (a) 10 units (b) 9 units

   (c) 12 units (d) 9/2 units

3. If the x-coordinate of a point A on the join of B(2, 2, 1) and C(5, 1, –2) is, then its z-coordinate is

   (a) –2 (b) –1

   (c) 1 (d) 2

4. The distance of the point M(a, b, c) from the x-axis is

   (a) $\sqrt{b^2+c^2}$ (b) $\sqrt{c^2+a^2}$

   (c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2+b^2+c^2}$

5. The straight line $\frac{x-3}{3}=\frac{y-2}{1}=\frac{z-1}{0}$ is

   (a) parallel to x-axis (b) parallel to y-axis

   (c) parallel to z-axis (d) perpendicular to z-axis

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. The shortest distance between the line $\frac{x-3}{3} = \frac{y}{0} = \frac{z}{-4}$ and y-axis is

   (a) $\frac{12}{5}$ units (b) $\frac{1}{5}$ units

   (c) 0 units (d) 3 units

2. The point of intersection of the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-1}{1} = \frac{y-2}{3} = \frac{z-6}{5}$ is

   (a) $\left(\frac{1}{3}, \frac{-1}{3}, -\frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

   (c) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$

3. If a line makes the same angle α, with each of the *x* and *z* axes and the angle β with y-axis such that $3\sin^2\alpha = \sin^2\beta$, then the value of $\cos^2\alpha$ is

   (a) $\frac{1}{5}$ (b) $\frac{2}{5}$

   (c) $\frac{3}{5}$ (d) $\frac{2}{3}$

4. If the lines $\frac{x+3}{k-5} = \frac{y-1}{1} = \frac{5-z}{-2k-1}$ and $\frac{x+2}{-1} = \frac{2-y}{-k} = \frac{z}{5}$ are perpendicular, then the value of *k* is

   (a) 1 (b) –1

   (c) 2 (d) –2

5. The image of the point P(1, 8, 4) to the line $\frac{x}{5} = \frac{y+1}{5} = \frac{z-3}{1}$ is

   (a) (5, 4, 4) (b) (5, 0, 4)

   (c) (9, 0, 4) (d) (1, 8, 4)

ANSWERS

ONE MARK QUESTIONS

1. $\sqrt{b^2 + c^2}$

2. 90°

3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$

4. $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + 3\hat{k})$

5. $\lambda = 2$

6. $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$

7. $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{2}{3}$

8. $\frac{x}{a} = \frac{y-1}{a} = \frac{z-2}{a}$, $a \in R - \{0\}$

9. $\frac{10}{\sqrt{14}}$

10. (–1, 2,–2)

11. $\vec{r} = \vec{0} + \lambda(6\hat{\imath} + 4\hat{\jmath} + 3\hat{k})$.

12. <0, 0, 1>, 0, 0, 1

13. <3, –1, 0>, $\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}$, 0

14. <a, 1, c>, $\vec{r} = b\hat{i} + d\hat{k} + \lambda(a\hat{i} + \hat{j} + c\hat{k})$

TWO MARK QUESTIONS

1. $\frac{x-2}{1} = \frac{y}{2} = \frac{z-5}{3}$

2. $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$

3. 2

4. $\vec{r} = (2\hat{i} + \hat{k}) + \lambda(2\hat{i} + 7\hat{j} - \hat{k})$

5. aa′ + cc′ + 1 = 0

6. .....

7. $\frac{x-2}{1} = \frac{y-1}{-7} = \frac{z-3}{2}$,
   $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(\hat{i} - 7\hat{j} + 2\hat{k})$

8. $\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6}$,
   $\vec{r} = (2\hat{i} + 3\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} - 6\hat{k})$

9. $\lambda = 4$
   $\mu = 3$

10. $\lambda = 10$

THREE/FIVE MARK QUESTIONS

1. $\bar{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

2. $\left(-\frac{51}{7}, -\frac{18}{7}, \frac{43}{7}\right)$

3. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

4. $\frac{1}{\sqrt{6}}$

5. $2\sqrt{29}$ units

6. $\frac{8}{\sqrt{29}}$

7. $(1, 2, 3), \sqrt{14}$

9. $SD = 14\, units,\ \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

11. K = $\frac{9}{2}$

12. $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

13. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

14. (1, 2, 3), $\sqrt{14}$, (0, 5, 1)

15. (6, 7, 3)

16. $\left(6, \frac{19}{2}, 18\right), \left(-2, \frac{-5}{2}, -6\right)$

17. $\frac{9}{\sqrt{195}}$, Not intersecting

18. (3, 5, 9), $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$, 7 units

19. $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$

20. $(-2, -1, 3), \left(\frac{56}{17}, \frac{43}{17}, \frac{11}{17}\right)$

21. $PQ : \vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$

$QR : \vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k})$, Point R(3,4,5)

22. $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$, (2,5,7), $\sqrt{13}$ units

23. (–1, –1, –1)

24. $\lambda \neq 3$

25. $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(2\hat{i} - \hat{j} + \hat{k})$, $\sqrt{\frac{11}{6}}$ units

SELF ASSESSMENT TEST-1

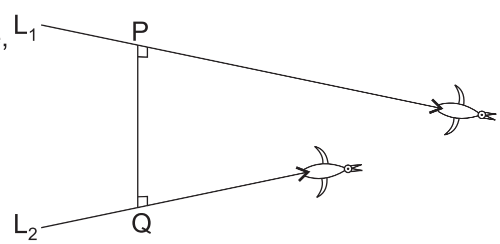
1. (C) 2. (B) 3. (B) 4. (A) 5. (D)

SELF ASSESSMENT TEST-2

1. (A) 2. (D) 3. (C) 4. (B) 5. (C)

Case Study Based Questions

1. Two birds are flying in the space along straight path L_1 and L_2 (Neither parallel nor intersecting) where,



$$L_1 = \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

$$L_2 = \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-3}{1}$$

On the basis of this answer the following

(a) Vector form of L_1 is

(i) $\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + \hat{j} + \hat{k})$

(ii) $r = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \lambda(3\hat{i} - \hat{j} - \hat{k})$

(iii) $\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$

(iv) $\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$

(b) If $PQ \perp L_1$ and $PQ \perp L_2$ and , then coordinates *P* are

(i) (3, 8, 3) (ii) (–3, 8, –3)

(iii) (3, –8, 3) (iv) (–3, –8, –3)

(c) Directions ratios of *PQ*

(i) 2, 1, 5 (ii) 2, 5, –1

(iii) 5, 1, 2 (iv) 5, 2, 1

(d) Distance *PQ* is

(i) $2\sqrt{15}$ units (ii) $2\sqrt{30}$ units

(iii) $3\sqrt{30}$ units (iv) $3\sqrt{20}$ units

(e) Equation of the path *PQ* is

(i) $r = (3i + 8j + 3k) + \lambda(2i + 5j - k)$

(ii) $r = (3i - 8j + 3k) + \lambda(2i + 5j - k)$

(iii) $r = (3i + 8j + 3k) + \lambda(2i - 5j + k)$

(iv) $r = (3i - 8j - 3k) + \lambda(2i - 5j - k)$

ANSWERS

1. (a) (iii) (b) (i) (c) (ii) (d) (iii) (e) (i)

CHAPTER-12

LINEAR PROGRAMMING

Linear programming is used to obtain optimal solutions for operations research. Using LPP, researchers find the **best**, most economical **solution** to a problem within all of its **limitations**, or constraints.

Few examples of applications of LPP

(i) **Food and Agriculture:** In nutrition, Linear programming provides a powerful tool to aid in planning for dietary needs. Here, we determine the different kinds of foods which should be included in a diet so as to **minimize** the cost of the desired diet such that it contains the minimum amount of each nutrient.

(ii) **Transportation:** Systems rely upon linear programming for cost and time efficiency.

Airlines use linear programming to optimize their profits according to different seat prices and customer demand. Because of this only, efficiency of airlines increases and expenses are decreased.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23
• Introduction, constraints, objective function, optimization.
• Graphical method of solution for problems in two variables.
• Feasible and infeasioble region (bounded or unbounded)
• Feasible and infeasible solutions.
• Optimal feasible solutions (upto three non-trival constraints)

KEY POINTS :

- **OPTIMISATION PROBLEM :** is a problem which seeks to maximize or minimize a function. An optimisation problem may involve maximization of profit, minimization of transportation cost etc, from available resources.
- **A LINEAR PROGRAMMING PROBLEM (LPP) :** LPP deals with the optimisation (maximisation/minimisation) of a linear function of two variables (say x and y) known as objective function subject to the conditions that the variables are non negative and satisfy a set of linear inequalities (called linear constraints). A LPP is a special type of optimisation problem.
- **OBJECTIVE FUNCTION :** Linear function $z = ax + by$ where a and b are constants which has to be maximised or minimised is called a linear objective function.
- **DECISION VARIABLES :** In the objective function $z = ax + by$, x and y are called decision variables.
- **CONSTRAINTS :** The linear inequalities or restrictions on the variables of an LPP are called constraints.

  The conditions $x \geq 0, y \geq 0$ are called non-negative constraints.
- **FEASIBLE REGION :** The common region determined by all the constraints including non-negative constraints $x \geq 0, y \geq 0$ of a LPP is called the feasible region for the problem.
- **FEASIBLE SOLUTION :** Points within and on the boundary of the feasible region for a LPP represent feasible solutions.
- **INFEASIBLE SOLUTIONS :** Any point outside the feasible region is called an infeasible solution.
- **OPTIMAL (FEASIBLE) SOLUTION :** Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
- **THEOREM 1 :** Let R be the feasible region (convex polygon) for a LPP and let $z = ax + by$ be the objective function. When z has an optimal value (maximum or minimum), where x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.
- **THEOREM 2 :** Let R be the feasible region for a LPP. & let $z = ax + by$ be the objective function. If R is bounded, then the objective function z has both a maximum and a minimum value on R and each of these occur at a corner point of R.

  If the feasible region R is unbounded, then a maximum or minimum value of the objective function may or not exist. However, if it exists it must occur at a corner point of R.

- **MULTIPLE OPTIMAL POINTS :** If two corner points of the feasible region are optimal solutions of the same type i.e both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Illustration:

A company produces two types of belts A and B. Profits on these belts are Rs. 2 and Rs. 1.50 per belt respectively. A belt of type A requires twice as much time as belt of type B. The company can produce atmost 1000 belts of type B per day. Material for 800 belts per day is available. Atmost 400 buckles for belts of type A and 700 for type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

Solution: Let the company produces x no. of belts of type A and y no. of belts of type B to maximize the profit.

$\therefore$ **Objective function** Max $z = 2x + 1.5y$

As, maximum 1000 belts of type B : 1 day

$\therefore$ 1 belt of type B : $\left(\frac{1}{1000}\right)^{th}$ of a day

ATQ, 1 belt of type A : $\left(\frac{2}{1000}\right)^{th}$ of a day

$\therefore \quad \frac{2x}{1000} + \frac{y}{1000} \leq 1$

$\Rightarrow \quad 2x + y \leq 1000$

L.P.P becomes

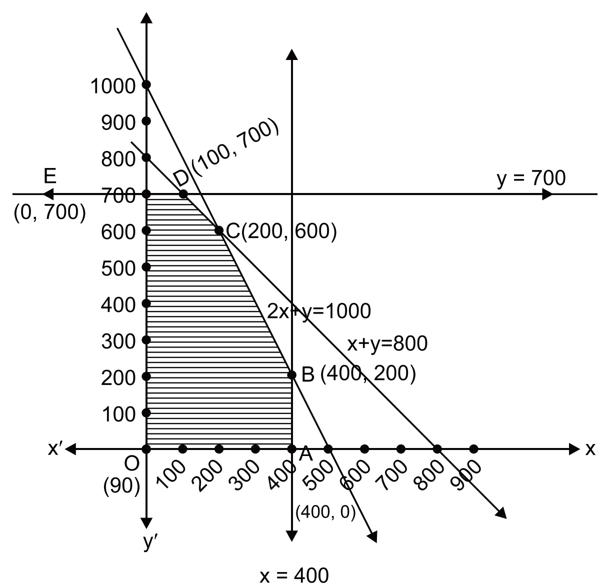
Max $z = 2x + 1.5y$

s.t. $2x + y \leq 1000$

$x + y \leq 800$

$x \leq 400,\ y \leq 700,\ x \geq 0,\ \geq 0$

Here, the feasible negion is bounded given be region OABCDE.



Using Corner point method.

Corner Points	Obj. fn. z = 2x + 1.5y
O (0, 0)	0
A (400, 0)	800
B (400, 200)	1100
C (200, 600)	1300 ← max z.
D (100, 700)	1250
E (0, 700)	1050

∴ Optimal solution is given by C(200, 600)

i.e. company should produce 200 belts of type A and 600 belts of type B so as to maximize the profit of Rs. 1300.

FIVE MARKS QUESTIONS

Q. 1 Solve the following LPP graphically.

Maximize $z = 3x + y$ subject to the constraints

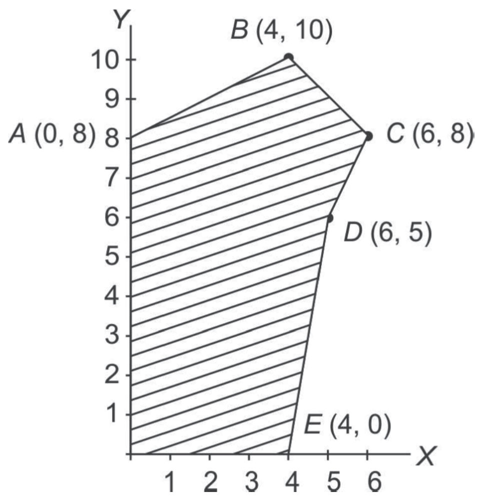
$x + 2y \geq 100$

$2x - y \leq 0$

$2x + y \leq 200$

$x, y \geq 0$

Q.2 The corner points of the feasible region determined by the system of linear constraints are as shown below.



Answer each of the following :

(i) Let $z = 3x - 4y$ be the objective function. Find the maximum and minimum value of z and also the corresponding points at which the maximum and minimum value occurs.

(ii) Let $z = px + qy$ where $p, q > 0$ be the objective function. FInd the condition on p and q so that the maximum value of z occurs at B (4, 10) and C (5, 8). Also mention the number of optimal solutions in this case.

Q. 3 There are two types of fertilisers A and B. A consists of 10% nitrogen and 6% phosphoric acid and B consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If A costs Rs. 6 per kg and B costs Rs. 5 per kg. determine how much of each type of fertiliser should be used so that nutrient requirements are met at minimum cost. What is the minimum cost?

Q. 4 A man has Rs. 1500 to purchase two types of of shares of two different companies S1 and S2. Market price of one share of S1 is Rs. 180 and S2 is Rs 120. He wishes to purchase a maximum of ten shares only. If one share of type S1 gives a yield of Rs 11, and of type S2 yields Rs 8 then how much shares of each type must be purchased to get maximum profit? and what will be the maximum profit?

Q. 5 A company manufactures two types of lamps say A and B. Both lamps go through a cutter and then a finisher. Lamp A requiires 2 hours of the cutter's time and 1 hour of the finisher's time. Lamp B required 1 hr of cutter's, 2 hrs of finisher's time. The cutter has 100 hours and finisher has 80 hours of time available each month. Profit on one lamp A is Rs. 7.00 and on one lamp B is Rs 13.00. Assuming that he can sell all that he produces how many of each type of lamps should be manufactured to obtain maximum profit and what will be the maximum profit?

Q.6 A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for atmost 20 items. A fan and sewing machine cost Rs 360 and Rs. 240 respectively. He can sell a fan at a profit of Rs. 22 and sewing machine at a profit of Rs. 18. Assuming that he can sell whatever he buys, how should he invest money to maximise his profit?

Q. 7 A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods X and Y. To produce one unit of X, 2 units of capital and 1 unit of labour is required. To produce one unit of Y, 3 units of labour and 1 unit of capital is required. If X and Y are priced at Rs. 80 and Rs100 per unit respectively, how should the producer use his resources to maximize revenue?

Q. 8 A factory owner purchases two types of machines A and B for his factory. The requirements and limitations for the machines are as follows :

Machine	Area Occupied	Labour Force	Daily Output (in units)
A	1000 m^2	12 men	50
B	1200 m^2	8 men	40

He has maximum area of 7600 m^2 available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

Q.9 A manufacturer makes two types of cups A and B. Three machines are required to manufacture the cups and the time in minutes required by each in as given below :

Types of Cup	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum period of 6 hours per day. If the profit on each cup A is 75 paisa and on B is 50 paisa, find how many cups of each type should be manufactured to maximise the profit per day.

Q. 10 An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and as profit of Rs. 300 is made on each second class ticket. The airline reserves atleast 20 seats for first class. However at least four times as many passengers prefer to travel by second class than by first class. Determine how many tickets of each type must be sold to maximize profit for the airline.

Q. 11 A diet for a sick person must contains at least 4000 units of vitamins, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of Rs. 5 and Rs. 4 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of minerals and 40 units of calories whereas one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the food A and B should be used to have least cost but it must satisfy the requirements of the sick person.

Q.12 Anil wants to invest at most Rs. 12000 in bonds A and B. According to the rules, he has to invest at least Rs. 2000 in Bond A and at least Rs. 4000 in bond B. If the rate of interest on bond A and B are 8% and 10% per annum respectively, how should he invest this money for maximum interest? Formute the problem as LPP and solve graphically.

ONE MARKS QUESTIONS

1. The feasible region for a LPP is always a _______ polygon.
2. A corner point of a feasible region is a point in the region which is the _______ of two boundary lines.
3. Regions represented by the equations $x \geq 0, y \geq 0$ is which quadrant?
4. Half plane below the x-axis including the points on x-axis is represented by which inequality?
5. State T/F

   The solution set of the inequation $2x + y > 5$ is open half plane not containing the origin.
6. What do we call a feasible region of a system of linear inequalities if it can be enclosed within a circle?
7. If in a LPP, the objective function $z = ax + by$ has the same maximum value on two corner points A & B of the feasible region, then how many optimal solutions will that LPP have?

8. What do we call the linear inequalities or restrictions on the variables in LPP?
9. When the optimal value of the objective function in a LPP may or may not exist.
10. If the feasible region of LPP is bounded, then name the method which is used to find the optimal solution.

CASE STUDY QUESTIONS

Q. 1 A man rides his motorcycle at the speed of 50 km/hr. He has to spend Rs 2/km on petrol. But if he rides it at a faster speed of 80 km/hr, the petrol cost increases to Rs 3/km. He has atmost Rs 120 to spend on petrol and one hr's time. he wishes to find the maximum distance that he can travel.

Based on the above information answer the following questions.

(1) If he travels x km with the speed of 50 km/hr and y km with the speed of 80 km/hr, then which of the following is false.

(a) Maximise $x + y$ (b) $\frac{x}{50} + \frac{y}{80} \leq 1$

(c) $\frac{x}{60} + \frac{y}{40} \leq 1$ (d) All of these

(2) Maximum distance man can travel is given by

(a) $54\frac{2}{7}$ km (b) 50 km

(c) 40 km (d) 52 km

(3) If he covers maximum distance then how much distance he travels with the speed of 50 km/hr.

(a) 50 km (b) 40 km

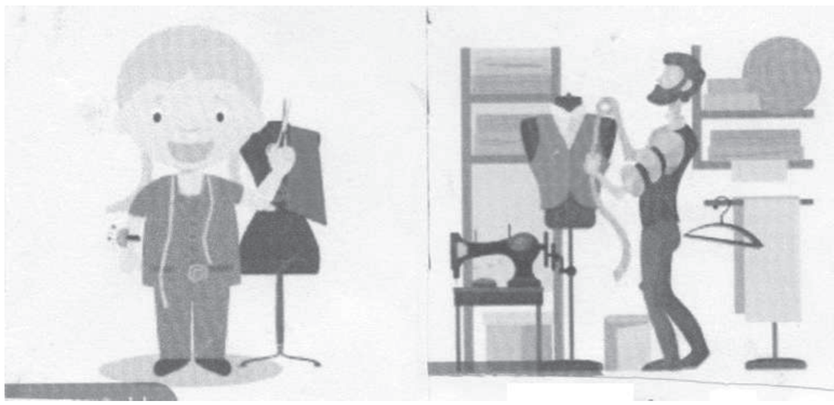
(c) $48\frac{6}{7}$ km (d) $11\frac{3}{7}$ km

(4) What is the average speed during the whole journey for covering the maximum distance.

(a) 80 km/hr (b) $55\frac{2}{7}$ km/hr

(c) $\frac{380}{7}$ km/hr (d) $53\frac{2}{7}$ km/hr

Q. 2 Two tailors A and B earn Rs 150 and Rs 200 per day respectively. A can sticth 6 shirts and 4 pants per day, while B can stitch 10 shirts and 4 pants per day. it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost.



Tailor A **Tailor B**

Based on the above information answer the following.

(1) If x and y are the number of days A and B work respectively then the objective function for this LPP is

(a) min $z = 150x + 200y$ (b) $6x + 10y \geq 60$

(c) $x + y \geq 8$ (d) $4x + 4y \geq 32$

(2) The optimal solution for this LPP is

(a) (10, 0) (b) (0, 8)

(c) (5, 3) (d) (0, 6)

(3) Minimum labour cost will be

(a) Rs 400 (b) Rs 1250

(c) Rs 1600 (d) Rs 1350

(4) In this LPP, feasible region is

(a) Bounded (b) Un bounded

(c) None of these (d) All of these

(5) In a LPP, the feasible region is always a _________ polygon.

(a) Convexo voncave (b) Concavo convex

(c) Concave (d) Convex

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Objective functin of a L.P.P. is
   - (a) A constraint
   - (b) A function of be optimised
   - (c) A relation between the variables
   - (d) None of these
2. The solution set of the inequation $2x + y > 5$ is
   - (a) Open half plans that contains the origion
   - (b) Open half plane not containing the origin
   - (c) Whole xy-plane except the points lying on the line $2x + y = 5$
   - (d) None of these
3. Which of the following statements is correct?
   - (a) Every L.P.P admits an optimal solution
   - (b) A L.P.P. admits unique optimal solution
   - (c) If a LPP admits two optimal solutions, it has an infinite number of optimal solutions
   - (d) None of these
4. Solution set of inequation $x \geq 0$ is
   - (a) Half plane on the left of y-axis
   - (b) Half plane on the right of y-axis excluding the points on y-axis
   - (c) Half plane on the right of y-axis including the points on y-axis
   - (d) None of these
5. In a LPP, the constraints on the decision variables x and y are

   $x - 3y \geq 0$, $y \geq 0$, $0 \leq x \leq 3$.

   The feasible region
   - (a) is not in the first quadrant
   - (b) is bounded in the first quadrant
   - (c) is unbounded in the first quadrant
   - (d) doesn't exist

SELF ASSESSMENT-2

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Solution set of the inequation $y \leq 0$ is
   (a) Half plane below the x-axis excluding the points on x-axis
   (b) Half plane below the x-axis including the points on x-axis
   (c) Half plane above the x-axis
   (d) None of these
2. Regions represented by inequations $x \geq 0$, $y \geq 0$ is
   (a) first quadrant
   (b) second quadrant
   (c) third quadrant
   (d) fourth quadrant
3. The feasible region for an LPP is always
   (a) concavo convex polygen
   (b) concave poloygon
   (c) convex polygon
   (d) None of these
4. If the constraints in a linear programming problem are changed then
   (a) the problem is to be reevaluated
   (b) solution not defined
   (c) the objective function has to be modified
   (d) the change in constraints is ignored
5. L.P.P. is as follows:

   Minimize $Z = 30x + 50y$

   Subject to the constraints,

   $3x + 5y \geq 15$

   $2x + 3y \leq 18$

   $x \geq 0, y \geq 0$

   In the feasible region, the minimum value of Z occurs at
   (a) a unique point
   (b) no point
   (c) infinitely many points
   (d) two points only

ANSWER

Five Marks Questions

1. Max $z = 250$ at $x = 50$, $y = 100$
2. (i) Max $z = 12$ at (4, 0) and min $z = -32$ at (0, 8)

   (ii) $p = q$, infinite solutions lying on the line segment joining the points B and C.
3. 100 kg of fertilizer A and 80 kg of fertilizer B, minimum cost Rs 1000
4. Maximum profit = Rs 95 with 5 shares of each type.
5. Lamps of type A = 40, Lamps of type B = 20 Max profit = Rs 540
6. Fans : 8, sewing machines : 12, max profit : Rs 392
7. X : 2 units, Y : 6 units, max revenue is Rs 760.
8. Type A : 4, Type B : 3
9. Cup A : 15, cup B : 30
10. No of first class ticket = 40, No of second class tickets = 160
11. Food A : 5 units, food B : 30 units
12. Maximum interest is Rs 1160 at (2000, 10000)

One Mark Questions

1. Convex	2. Intersection	3. Ist quadrant
4. $y \leq 0$	5. True	6. Bounded
7. Infinite	8. LInear constraints	

9. If the feasible region is unbounded
10. Corner point method

CASE STUDIES QUESTIONS

1. (i) (d)	(ii) (a)	(iii) (c)	(iv) (c)	
2. (i) (a)	(ii) (c)	(iii) (d)	(iv) (d)	(v) (d)

SELF ASSESSMENT-1

1. (b)	2. (b)	3. (c)	4. (c)	5. (b)

SELF ASSESSMENT-2

1. (b)	2. (a)	3. (c)	4. (a)	5. (c)

CHAPTER-13

PROBABILITY

Probability is the branch of mathematics that deals with assigning a numerical quantity ($0 \leq p \leq 1$) to the happenning/non happenning of any event.

Real Life Applications
- Weather Forecasting
- Sports betting
- Investing in stock market
- Politics
- How likely a natural disaster like earthquake, hurricane etc. will strike the country in a given year.

A sports betting company may look at the current record of two teams A and B and determine which team has higher probability of winning and do the sports betting accordingly.

TOPICS TO BE COVERED AS PER CBSE LATEST CURRICULUM 2022-23

- Conditional probability
- Multiplication theorem on probability
- Independent events
- Total probability and Baye's theorem
- Random variable and its probability distribution
- Mean of random variable

KEY POINTS

Conditional Probability : If A and B are two events associated with the same sample space of a random experiment, then the conditional probability of the event A under the condition that the event B has already occurred, written as $P(A|B)$, is given by

$$P(A|\text{B}) = \frac{(P \cap B)}{P(B)}, \quad P(B) \neq 0.$$

Properties :

(1) $P(S|F) = P(F|F) = 1$ where S denotes sample space

(2) $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$

(3) $P(E'|F) = 1 - P(E|F)$

Multiplication Rule : Let E and F be two events associated with a sample place of an experiment. Then

$$P(E \cap F) = P(E)\,P(F|E) \text{ provided } P(E) \neq 0$$

$$= P(F)\,P(E|F) \text{ provided } P(F) \neq 0.$$

If E, F, G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E)\,P(F|E)\,P(G|(E \cap F))$$

Independent Events : Let E and F be two events, then if probability of one of them is not affected by the occurrence of the other, then E and F are said to be independent, i.e.,

(a) $P(F|E) = P(F)$, $P(E) \neq 0$

or (b) $P(E|F) = P(E)$, $P(F) \neq 0$

or (c) $P(E \cap F) = P(E)\,P(F)$

Three events A, B, C are mutually independent if

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

$$P(A \cap B) = P(A)\,P(B)$$

$$P(B \cap C) = P(B)\,P(C)$$

and $P(A \cap C) = P(A)\,P(C)$

Partition of a Sample Space : A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of a sample space S if

(a) $E_i \cap E_j = \phi$; $i \neq j$; $i, j = 1, 2, 3, ..., n$

(b)$E_1 \cup E_2 \cup E_3 ... \cup E_n = S$ and

(c) Each $E_i \neq \phi$ i.e. $P(E_i) > 0 \ \forall \ i = 1, 2, ..., n$

Theorem of Total Probability : Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S. Let A be the any event associated with S, then

$$P(A) = \sum_{j=1}^{n} P(E_j)\, P(A|E_j)$$

Baye's Theorem : If $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive events associated with a sample space S, and A is any event associated with E_i's having non-zero probability, then

$$P(E_i|A) = \frac{P(A \mid E_i)P(E_i)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}$$

Random Variable : A (r.v.) is a real variable which is associated with the outcome of a random experiment.

Probability Distribution of a r.v. X is the system of numbers given by

X :	x_1	x_2	....	x_n
$P(X = x)$:	p_1	p_2	....	p_n

where $\quad p_i > 0, \quad i = 1, 2, ..., n, \ \sum_{i=1}^{n} p_i = 1.$

Mean of a r.v. *X* :

$$\mu = E(X) = \sum_{i=1}^{n} p_i\, x_i$$

Illustration:

Evaluate $P(A \cup B)$ if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

Solution: $2P(A) = P(B) = \frac{5}{13}$

$\Rightarrow \ P(A) = \frac{5}{26}, \ P(B) = \frac{5}{13}$

As $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow \frac{2}{5} = \frac{P(A \cap B)}{(5/13)} \quad \Rightarrow \quad \frac{2}{\cancel{5}} \times \frac{\cancel{5}}{13} = P(A \cap B)$$

$$\Rightarrow \frac{2}{13} = P(A \cap B)$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{16}$$

Illustration:

Prove that if E and F are independent events, then the events E and F′ are also independent.

Solution: $P(E \cap F) = P(E)\,P(F)$ (given)

Consider, $P(E \cap F') = P(E) - P(E \cap F)$

$= P(E) - P(E)\,P(F)$

$= P(E)\,(1 - P(F))$

$P(E \cap F') = P(E) - F(F')$

So, E and F′ are also independent.

Illustration:

A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds.

Solution: Let E_1 = lost card is diamond

E_2 = lost card is non-diamond

A = 2 diamonds cards are drawn from the remaining cards

Using Theorem of total probability

$$P(A) = P(A|E_1)\,P(E_1) + P(A|E_2)\,P(E_2)$$

$$= \frac{12}{51} \times \frac{11}{50} \times \frac{\cancel{13}}{\cancel{52}_4} + \frac{12}{50} \times \frac{13}{51} \times \frac{\cancel{39}^3}{\cancel{52}_4}$$

$$= \frac{132}{10200} + \frac{468}{10200} = \frac{\cancel{600}}{\cancel{10200}_{17}} = \frac{1}{17}$$

Illustration:

Three cards are drawn at random (without replacement) from a well shuffled pack of 52 playing cards. Find the probability distribution of number of red cards. Hence, find the mean of the distribution.

Solution: Let X denotes the number of red cards

$$\therefore \quad P(X = 0) = \frac{{}^{26}C_3}{{}^{52}C_3} = \frac{26 \times 25 \times 24}{52 \times 51 \times 50} = \frac{2}{17} = \frac{4}{34}$$

$$P(X = 1) = \frac{{}^{26}C_1 \times {}^{26}C_2}{{}^{52}C_3} = 26 \times \frac{26 \times 25}{2} \times \frac{3 \times 2 \times 1}{52 \times 51 \times 50} = \frac{13}{34}$$

$$P(X = 2) = \frac{{}^{26}C_2 \times {}^{26}C_1}{{}^{52}C_3} = \frac{26 \times 25 \times 26 \times 3 \times 2 \times 1}{2 \times 52 \times 51 \times 50} = \frac{13}{34}$$

$$P(X = 3) = \frac{{}^{26}C_3}{{}^{52}C_3} = \frac{4}{34}$$

$\therefore$ **Probability Distribution**

X	P(X = x)	X.P(x)
0	$\frac{4}{34}$	0
1	$\frac{13}{34}$	$\frac{13}{34}$
2	$\frac{13}{34}$	$\frac{26}{34}$
3	$\frac{4}{34}$	$\frac{12}{34}$
	$\sum p_i = 1$	$\bar{x} = \sum p_i x_i$

$$\therefore \quad \bar{x} = \sum p_i x_i = \frac{13}{34} + \frac{26}{34} + \frac{12}{34} = \frac{51}{34} = \frac{3}{2}$$

ONE MARK QUESTIONS

1. The probabilities of A and B solving a problem independently are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. If both of them try to solve the problem independently, what is the probability that the problem is solved ?

2. The probability that it will rain on any particular day is 50%. Find the probability that it rains only on first 4 days of the week.

3. Write the value of $P(A|B)$ if $P(A) = 0.4$, $P(B) = 0.8$ and $P(B|A) = 0.6$.

4. A soldier fires three bullets on enemy. The probability that the enemy will be killed by one bullet is 0.7. What is the probability that the enemy is still alive ?

5. If $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\bar{A} \text{ or } \bar{B}) = \frac{1}{4}$. State whether A and B are independent.

6. A natural number x is chosen at random from the first hundred natural numbers. Find the probability such that $x + \frac{1}{x} < 2$.

7. A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). What is the probability that $x_3 = 30$.

8. Let A and B be two events such that $P(A) = 0.6$, $P(B) = 0.2$ and $P(A|B) = 0.5$. Then find $P(A'|B')$.

9. The probability that a person is not a swimmer is 0.3. Find the probability that out of 5 persons 4 are swimmers.

10. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, then find the probability that both are dead.

TWO MARKS QUESTIONS

1. A and B are two events such that $P(A) \neq 0$, then find $P(B|A)$ if (*i*) A is a subset of B (*ii*) $A \cap B = \phi$.

2. A random variable X has the following probability distribution, find k.

X	0	1	2	3	4	5
$P(X)$	$\frac{1}{15}$	K	$\frac{15K-2}{15}$	K	$\frac{15K-1}{15}$	$\frac{1}{15}$

3. Out of 30 consecutive integers two are chosen at random. Find the probability so that their sum is odd.

4. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. Find the probability that the eldest child is a girl given that the family has atleast one girl.

5. If A and B are such that $P(A \cup B) = \frac{5}{9}$ and $P(\bar{A} \cup \bar{B}) = \frac{2}{3}$, then find $P(\bar{A}) + P(\bar{B})$.

6. Prove that if A and B are independent events, then A and B' are also independent events.

7. If A and B are two independent events such that $P(A) = 0.3$, $P(A \cup B) = 0.5$, then find $P(A|B) - P(B|A)$

8. Three faces of an ordinary dice are yellow, two faces are red and one face is blue. The dice is rolled 3 times. Find the probability that yellow, red and blue face appear in the first, second and third throw respectively.

9. Find the probability that a leap year will have 53 Fridays or 53 Saturdays.

10. A person writes 4 letters and addresses on 4 envelopes. If the letters are placed in the envelopes at random, then what is the probability that all the letters are not placed in the right envelopes.

11. Find the mean of the distribution

X = x	0	1	2	3	4	5
P(X = x)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

12. In a class XII of a school, 40% of students study Mathematics, 30% of the students study Biology and 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, then find the probability that he will be studying Mathematics or Biology.

THREE MARKS QUESTIONS

Q.1. A problem in mathematics is given to three students whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem is solved ?

Q.2. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$ then find $P(A)$ and $P(B)$.

Q.3. From a lot of 20 bulbs which include 5 defectives, a sample of 2 bulbs is drawn at random, one by one with replacement. Find the probability distribution of the number of defective bulbs. Also, find the mean of the distribution.

Q.4. Amit and Nisha appear for an interview for two vacancies in a company. The probability of Amit's selection is 1/5 and that of Nisha's selections is 1/6. What is the probability that

(i) both of them are selected?

(ii) only one of them is selected?

(iii) none of them is selected?

Q.5. In a game, a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.

Q.6. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability that the selected person is male assuming that there are 60% males and 40% females ?

Q.7. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.

Q.8. Two aeroplanes X and Y bomb a target in succession. There probabilities to hit correctly are 0.3 and 0.2 respectively. The second plane will bomb only if first miss the target. Find the probability that target is hit by Y plane.

Q.9. The random variable X can take only the values 0, 1, 2. Given that $P(X = 0) = P(X = 1) = p$ and that $E(X^2) = E(X)$, find the value of p.

Q.10. An urn contains 4 white and 3 red balls. Let X be the number of red balls in a random draw of 3 balls. Find the mean of X.

FIVE MARKS QUESTIONS

Q.1. By examining the chest X-ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of a healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB ?

Q.2. Three persons A, B and C apply for a job of Manager in a private company. Chances of their selection (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce charges to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change doesn't take place, find the probability that it is due to the appointment of C.

Q.3. A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letters TA are visible. What is the probability that the letter came from TATANAGAR.

Q.4. [illegible] X is given as under :

$$P(X = x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{Otherwise} \end{cases}$$

where k is a constant. Calculate

(*i*) $E(X)$ (*ii*) $E(3X^2)$ (*iii*) $P(X \geq 4)$

Q.5. Three critics review a book. Odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4 respectively for the three critics. Find the probability that the majority are in favour of the book.

Q.6. Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6, 7. Let X denotes the larger of the two numbers obtained. Find the mean of the probability distribution of X.

Q.7. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white?

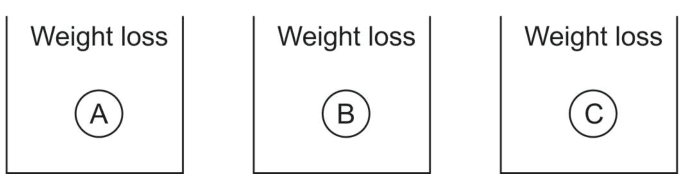
Q.8. Two cards are drawn from a well shuffled pack of 52 cards. Find the mean and variance for the number of face cards obtained.

Q.9. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to be both clubs. Find the possibility of the lost card being of club.

Q.10. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II at random. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

CASE STUDY QUESTIONS

Q.1. A company sells three types of Nutritional foods A, B, C for a weightloss programme. These are sold as a mixture where the proportions are 4 : 4 : 2 respectively. The probability of loosing weight by these foods A, B and C are 75%, 80% and 60% respectively.



Based on the above information, answer the following questions :

(*a*) Calculate the probability of randomly chosen food to do weight loss.

(*i*) 73% (*ii*) 74% (*iii*) 75% (*iv*) 95%

(*b*) Calculate the probability that there is no weightloss when it is given that the person takes food *B*

(*i*) 80% (*ii*) 60% (*iii*) 20% (*iv*) 40%

(*c*) Calculate the probability that food was of type *C* given that there is reduction in weight.

(*i*) $\frac{32}{74}$ (*ii*) $\frac{12}{74}$ (*iii*) $\frac{8}{74}$ (*iv*) $\frac{74}{16}$

(*d*) The probability that there is no reduction in weight given the food *C* is

(*i*) 0.4 (*ii*) 0.6 (*iii*) 0.25 (*iv*) 0.2

(*e*) What is the probability of choosing food *B* given that there is no weight loss.

(*i*) 0.4 (*ii*) 0.8 (*iii*) 4/25 (*iv*) 0.3

Q.2. In a birthday party, a magician was being invited by a parent and he had 3 bags that contain number of red and white balls as follows :

Bag 1 : 3 red balls, Bag 2 : 2 white balls and 1 red ball

Bag 3 : 3 white balls

The probability that the bag *i* will be chosen by the magician and a ball is selected from it is $\frac{i}{6}$, $i = 1, 2, 3$.

Based on the above information, answer the following questions.

(*a*) What is the probability that a red ball is selected by the magician

(*i*) $\frac{13}{18}$ (*ii*) $\frac{5}{6}$ (*iii*) $\frac{1}{6}$ (*iv*) $\frac{5}{18}$

(*b*) What is the probability that a white ball is selected by the magician

(*i*) $\frac{5}{6}$ (*ii*) $\frac{13}{18}$ (*iii*) $\frac{1}{6}$ (*iv*) $\frac{5}{18}$

(*c*) Given that the magician selects the white balls, what is the probability that this ball was from Bag 2.

(*i*) $\frac{4}{13}$ (*ii*) $\frac{4}{5}$ (*iii*) 0 (*iv*) $\frac{1}{2}$

(*d*) Given that the magician selects the red ball, what is the probability that this ball was from Bag 1.

(*i*) $\frac{3}{5}$ (*ii*) $\frac{4}{5}$ (*iii*) $\frac{4}{13}$ (*iv*) $\frac{1}{2}$

(*e*) What is the probability of selecting either red or white ball from Bag 2.

(*i*) 0 (*ii*) $\frac{4}{5}$ (*iii*) $\frac{1}{5}$ (*iv*) 1

Q.3. In an office three employees Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay process 50% of the forms. Sonia processes 20% and Iqbal the remaining 30% of the forms. Vinay has an error rate of 0.06, Sonia has an error rate of 0.04 and Iqbal has an error rate of 0.03.

Based on the above information answer the following :

(*i*) The conditional probability that an error is committed in processing given that Sonia processed the form is :

(*a*) 0.0210 (*b*) 0.04 (*c*) 0.47 (*d*) 0.06

(*ii*) The probability that Sonia processed the form and committed an error is :

(*a*) 0.005 (*b*) 0.006 (*c*) 0.008 (*d*) 0.68

(*iii*) The total probability of committing an error in processing the form is

(*a*) 0 (*b*) 0.047 (*c*) 0.234 (*d*) 1

(*iv*) The manger of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected has an error, the probability that the form is NOT processed by Vinay is :

(*a*) 1 (*b*) 30/47 (*c*) 20/47 (*d*) 17/47 (*v*) Let A be the event of committing an error in processing the form and let E_1, E_2 and E_3 be the events that Vinay, Sonia and Iqbal processed the form. The value of $\sum_{i=1}^{3} P(E_i \mid A)$ is

(*a*) 0 (*b*) 0.03 (*c*) 0.06 (*d*) 1

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If A and B are independent events such that P(A) = 0.4, P(B) = x and P(A $\cup$ B) = 0.5, then x = ?

   (a) $\frac{4}{5}$ (b) 0.1

   (c) $\frac{1}{6}$ (d) None of these

2. If P(A) = 0.8, P(B) = 0.5 and P(B/A) = 0.4, then P(A/B) is

   (a) 0.32 (b) 0.64

   (c) 0.16 (d) 0.25

3. A couple has two children. What is the probability that both are boys if it is known that one of them is a boy?

   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$

   (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

4. The random variable X has a probability distribution P(X) of the following form, where 'k' is some number.

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

   Determine the value of k.

5. If two events are independent, then
   (a) they must be mutually exclusive
   (b) the sum of their probabilities must be equal to 1
   (c) (a) and (b) both are correct
   (d) none of the above is correct

SELF ASSESSMENT-2

EAH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability that both cards are queens is

   (a) $\frac{1}{13}\times\frac{1}{13}$ (b) $\frac{1}{13}+\frac{1}{13}$

   (c) $\frac{1}{13}\times\frac{1}{17}$ (d) $\frac{1}{13}\times\frac{4}{51}$

2. The probability distribution of a discrete random variable X is given below:

X	2	3	4	5
P(X = x)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

   The values of k is
   (a) 8 (b) 16
   (c) 32 (d) 48

3. Three persons A, B and C fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is
   (a) 0.024 (b) 0.188
   (c) 0.336 (d) 0.452

4. If $4P(A) = 6P(B) = 10P(A \cap B) = 1$, then $P(B/A) = ?$

   (a) $\frac{2}{5}$ (b) $\frac{3}{5}$

   (c) $\frac{7}{10}$ (d) $\frac{19}{60}$

5. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

   (a) $\frac{5}{17}$ (b) $\frac{12}{17}$

   (c) $\frac{17}{30}$ (d) $\frac{3}{5}$

ANSWER

One Mark Questions

1. $\frac{1}{2}$
2. $\left(\frac{1}{2}\right)^7$
3. 0.3
4. $(0.3)^3$
5. No
6. 0
7. $\frac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5}$
8. $\frac{3}{8}$
9. 0.36
10. $\frac{3}{28}$

Two Marks Questions

1. (*i*) 1 (*ii*) 0
2. $\frac{4}{15}$
3. $\frac{15}{29}$
4. $\frac{4}{7}$
5. $\frac{10}{9}$
7. $\frac{1}{70}$
8. $\frac{1}{36}$
9. $\frac{3}{7}$
10. $\frac{23}{24}$
11. $\frac{35}{18}$
12. 0.6

Three Marks Questions

1. $\frac{3}{4}$
2. $P(A) = \frac{1}{5}$ and $P(B) = \frac{1}{6}$ or $P(A) = \frac{5}{6}$ and $P(B) = \frac{4}{5}$
3. $\frac{1}{2}$
4. (i) $\frac{1}{30}$ (ii) $\frac{3}{10}$ (iii) $\frac{2}{3}$
5. $-\frac{91}{54}$
6. $\frac{3}{4}$
7. $\frac{5}{9}$
8. $\frac{7}{22}$
9. $\frac{1}{2}$
10. $\frac{9}{7}$

Five Marks Questions

1. $\frac{110}{221}$
2. $\frac{7}{10}$
3. $\frac{7}{11}$
4. (*i*) 4.31, (*ii*) 61.9, (*iii*) $\frac{15}{22}$
5. $\frac{209}{343}$
6. $\bar{x} = \frac{17}{3}$
7. $\frac{1}{2}$
8. $\bar{x} = \frac{6}{13}$, $\sigma^2 = \frac{60}{169}$
9. $\frac{11}{50}$
10. $\frac{16}{31}$

CASE STUDY QUESTIONS

1. (*a*) (*ii*)	(*b*) (*iii*)	(*c*) (*ii*)	(*d*) (*i*)	(*e*) (*iv*)
2. (*a*) (*iv*)	(*b*) (*ii*)	(*c*) (*i*)	(*d*) (*i*)	(*e*) (*iv*)
3. (*i*) (*b*)	(*ii*) (*c*)	(*iii*) (*b*)	(*iv*) (*d*)	(*v*) (*d*)

SELF ASSESSMENT-1

1. (c)	2. (b)	3. (a)	4. $k = \frac{1}{6}$	5. (d)

SELF ASSESSMENT-2

1. (a)	2. (c)	3. (b)	4. (a)	5. (b)

Sample Question Paper
Class XII
Session 2022-23
Mathematics (Code-041)

Time Allowed: 3 Hours **Maximum Marks: 80**

General Instructions :

1. This Question paper contains - **five sections** A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. **Section A** has 18 **MCQ's and 02** Assertion-Reason based questions of 1 mark each.
3. **Section B** has 5 **Very Short Answer (VSA)-type** questions of 2 marks each.
4. **Section C** has 6 **Short Answer (SA)-type** questions of 3 marks each.
5. **Section D** has 4 **Long Answer (LA)-type** questions of 5 marks each.
6. **Section E** has 3 **source based/case based/passage based/integrated units of assessment (**4 marks each**)** with sub parts.

SECTION A
(Multiple Choice Questions)
Each question carries 1 mark

Q1. If A =[a_{ij}] is a skew-symmetric matrix of order n, then
(a) $a_{ij} = \frac{1}{a_{ji}} \forall\, i, j$ (b) $a_{ij} \neq 0 \,\forall\, i, j$ (c)$a_{ij} = 0, where\ i = j$ (d) $a_{ij} \neq 0\ where\ i = j$

Q2. If A is a square matrix of order 3, $|A'| = -3$, then $|AA'| =$
(a) 9 (b) -9 (c) 3 (d) -3

Q3. The area of a triangle with vertices A, B, C is given by
(a) $|\overrightarrow{AB} \times \overrightarrow{AC}|$ (b) $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$
(b) $\frac{1}{4}|\overrightarrow{AC} \times \overrightarrow{AB}|$ (d) $\frac{1}{8}|\overrightarrow{AC} \times \overrightarrow{AB}|$

Q4. The value of 'k' for which the function f(x) $= \begin{cases} \frac{1-cos4x}{8x^2}, if\ x \neq 0 \\ k, if\ x = 0 \end{cases}$ is continuous at x = 0 is
(a) 0 (b) -1 (c) 1. (d) 2

Q5. If $f'(x) = x + \frac{1}{x}$, then $f(x)$ is
(a) $x^2 + \log\,|x| + C$ (b) $\frac{x^2}{2} + \log\,|x| + C$ (c)$\frac{x}{2} + \log\,|x| + C$ (d) $\frac{x}{2} - \log\,|x| + C$

Q6. If m and n, respectively, are the order and the degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4 = 0$, then m + n =
(a) 1 (b) 2 (c) 3 (d) 4

Q7. The solution set of the inequality 3x + 5y < 4 is
(a) an open half-plane not containing the origin.
(b) an open half-plane containing the origin.
(c) the whole *XY*-plane not containing the line 3x + 5y = 4.
(d) a closed half plane containing the origin.

Q8. The scalar projection of the vector $3\hat{\imath} - \hat{\jmath} - 2\hat{k}$ *on the vector* $\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ is

(a) $\frac{7}{\sqrt{14}}$ (b) $\frac{7}{14}$ (c) $\frac{6}{13}$ (d) $\frac{7}{2}$

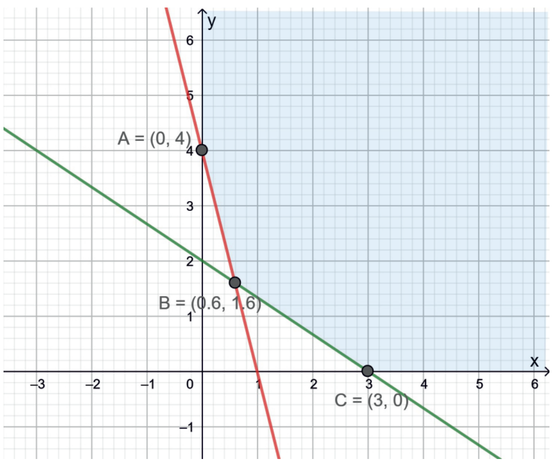
Q9. The value of $\int_2^3 \frac{x}{x^2+1} dx$ is

(a) log4 (b) $log\frac{3}{2}$ (c) $\frac{1}{2}log2$ (d) $log\frac{9}{4}$

Q10. If A, B are non-singular square matrices of the same order, then $(AB^{-1})^{-1} =$

(a) $A^{-1}B$ (b) $A^{-1}B^{-1}$ (c) BA^{-1} (d) AB

Q11. The corner points of the shaded unbounded feasible region of an LPP are (0, 4), (0.6, 1.6) and (3, 0) as shown in the figure. The minimum value of the objective function Z = 4x + 6y occurs at



(a) $(0.6, 1.6)$ *only* (b) (3, 0) only (c) (0.6, 1.6) and (3, 0) only

(d) at every point of the line-segment joining the points (0.6, 1.6) and (3, 0)

Q12. If $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$, *then the possible* value(s) of 'x' is/are

(a) 3 (b) $\sqrt{3}$ (c) $-\sqrt{3}$ (d) $\sqrt{3}, -\sqrt{3}$

Q13. If A is a square matrix of order 3 and |A| = 5, then $|adjA| =$

(a) 5 (b) 25 (c) 125 (d) $\frac{1}{5}$

Q14. Given two independent events A and B such that P(A) =0.3, P(B) = 0.6 and P$(A' \cap B')$ is

(a) 0.9 (b) 0.18 (c) 0.28 (d) 0.1

Q15. The general solution of the differential equation $ydx - xdy = 0$ *is*

(a) $xy = C$ (b) $x = Cy^2$ (c) $y = Cx$ (d) $y = Cx^2$

Q16. If $y = sin^{-1}x$, then $(1 - x^2)y_2$ *is* equal to

(a) xy_1 (b) xy (c) xy_2 (d) x^2

Q17. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}.\vec{b} = 4$, then $|\vec{a} - 2\vec{b}|$ is equal to

(a) $\sqrt{2}$ (b) $2\sqrt{6}$ (c) 24 (d) $2\sqrt{2}$

Q18. P is a point on the line joining the points $A(0,5,-2)$ and $B(3,-1,2)$. If the x-coordinate of P is 6, then its z-coordinate is

(a) 10 (b) 6 (c) -6 (d) -10

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

Q19. **Assertion (A)**: The domain of the function $sec^{-1}2x$ is $\left(-\infty, -\frac{1}{2}\right] \cup [\frac{1}{2}, \infty)$

Reason (R): $sec^{-1}(-2) = -\frac{\pi}{4}$

Q20. **Assertion (A)**: The acute angle between the line $\bar{r} = \hat{\imath} + \hat{\jmath} + 2\hat{k} + \lambda(\hat{\imath} - \hat{\jmath})$ and the x-axis is $\frac{\pi}{4}$

Reason(R): The acute angle θ between the lines

$\bar{r} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k})$ and

$\bar{r} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} + \mu(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k})$ is given by $cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$

SECTION B

This section comprises of very short answer type-questions (VSA) of 2 marks each

Q21. Find the value of $sin^{-1}[sin\left(\frac{13\pi}{7}\right)]$

OR

Prove that the function f is surjective, where $f: N \to N$ such that

$$f(n) = \begin{cases} \frac{n+1}{2}, if\ n\ is\ odd \\ \frac{n}{2}, if\ n\ is\ even \end{cases}$$

Is the function injective? Justify your answer.

Q22. A man 1.6 m tall walks at the rate of 0.3 m/sec away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening?

Q23. If $\vec{a} = \hat{\imath} - \hat{\jmath} + 7\hat{k}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath} + \lambda\hat{k}$, then find the value of λ so that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

OR

Find the direction ratio and direction cosines of a line parallel to the line whose equations are

$6x - 12 = 3y + 9 = 2z - 2$

Q24. If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, *then prove that* $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$

Q25. Find $|\vec{x}|$ if $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$, where $\vec{a}$ is a unit vector.

SECTION C

(This section comprises of short answer type questions (SA) of 3 marks each)

Q26. Find: $\int \frac{dx}{\sqrt{3-2x-x^2}}$

Q27. Three friends go for coffee. They decide who will pay the bill, by each tossing a coin and then letting the "odd person" pay. There is no odd person if all three tosses produce the same result. If there is no odd person in the first round, they make a second round of tosses and they continue to do so until there is an odd person. What is the probability that exactly three rounds of tosses are made?

OR

Find the mean number of defective items in a sample of two items drawn one-by-one without replacement from an urn containing 6 items, which include 2 defective items. Assume that the items are identical in shape and size.

Q28. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tanx}}$

OR

Evaluate: $\int_0^4 |x-1|\, dx$

Q29. Solve the differential equation: $ydx + (x - y^2)dy = 0$

OR

Solve the differential equation: $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

Q30. Solve the following Linear Programming Problem graphically:

Maximize Z = 400x + 300y subject to $x + y \leq 200, x \leq 40, x \geq 20, y \geq 0$

Q31. Find $\int \frac{(x^3+x+1)}{(x^2-1)} dx$

SECTION D

(This section comprises of long answer-type questions (LA) of 5 marks each)

Q32. Make a rough sketch of the region $\{(x, y): 0 \leq y \leq x^2, 0 \leq y \leq x, 0 \leq x \leq 2\}$ and find the area of the region using integration.

Q33. Define the relation R in the set $N \times N$ as follows:

For (a, b), (c, d) ∈ $N \times N$, (a, b) R (c, d) iff ad = bc. Prove that R is an equivalence relation in $N \times N$.

OR

Given a non-empty set X, define the relation R in P(X) as follows:
For A, B $\in P(X), (A, B) \in R$ iff $A \subset B$. Prove that R is reflexive, transitive and not symmetric.

Q34. An insect is crawling along the line $\bar{r} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \lambda(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$ and another insect is crawling along the line $\bar{r} = -4\hat{\imath} - \hat{k} + \mu(3\hat{\imath} - 2\hat{\jmath} - 2\hat{k})$. At what points on the lines should they reach so that the distance between them is the shortest? Find the shortest possible distance between them.

OR

The equations of motion of a rocket are:
$x = 2t, y = -4t, z = 4t,$ where the time t is given in seconds, and the coordinates of a moving point in km. What is the path of the rocket? At what distances will the rocket be from the starting point O(0, 0, 0) and from the following line in 10 seconds?
$\vec{r} = 20\hat{\imath} - 10\hat{\jmath} + 40\hat{k} + \mu(10\hat{\imath} - 20\hat{\jmath} + 10\hat{k})$

Q35. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use A^{-1} to solve the following system of equations
$2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3$

SECTION E

(This section comprises of 3 case-study/passage-based questions of 4 marks each with two sub-parts. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.)

Q36. **Case-Study 1**: Read the following passage and answer the questions given below.

The temperature of a person during an intestinal illness is given by
$f(x) = -0.1x^2 + mx + 98.6, 0 \leq x \leq 12$, m being a constant, where f(x) is the temperature in °F at x days.

(i) Is the function differentiable in the interval (0, 12)? Justify your answer.
(ii) If 6 is the critical point of the function, then find the value of the constant m.

(iii) Find the intervals in which the function is strictly increasing/strictly decreasing.

OR

(iii) Find the points of local maximum/local minimum, if any, in the interval (0, 12) as well as the points of absolute maximum/absolute minimum in the interval [0, 12]. Also, find the corresponding local maximum/local minimum and the absolute maximum/absolute minimum values of the function.

Q37. **Case-Study** 2: Read the following passage and answer the questions given below.

In an elliptical sport field the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) If the length and the breadth of the rectangular field be 2x and 2y respectively, then find the area function in terms of x.

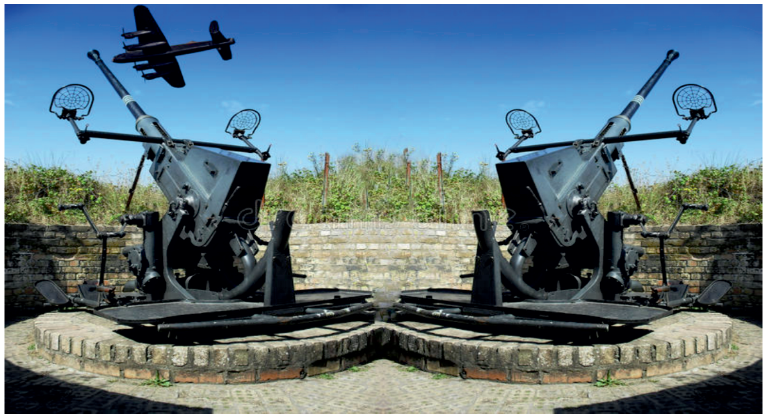
(ii) Find the critical point of the function.

(iii) Use First derivative Test to find the length 2x and width 2y of the soccer field (in terms of a and b) that maximize its area.

OR

(iii) Use Second Derivative Test to find the length 2x and width 2y of the soccer field (in terms of a and b) that maximize its area.

Q38. **Case-Study** 3: Read the following passage and answer the questions given below.



There are two antiaircraft guns, named as A and B. The probabilities that the shell fired from them hits an airplane are 0.3 and 0.2 respectively. Both of them fired one shell at an airplane at the same time.

(i) What is the probability that the shell fired from exactly one of them hit the plane?

(ii) If it is known that the shell fired from exactly one of them hit the plane, then what is the probability that it was fired from B?

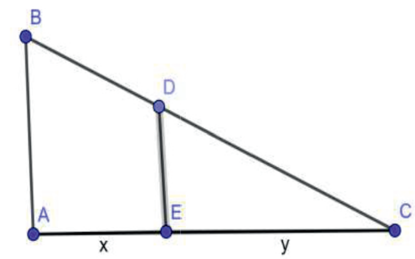
Marking Scheme
Class XII
Mathematics (Code – 041)
Section : A (Multiple Choice Questions- 1 Mark each)

Question No	**Answer**	**Hints/Solution**
1.	(c)	In a skew-symmetric matrix, the (i, j)th element is negative of the (j, i)th element. Hence, the (i, i)th element = 0
2.	(a)	$\|AA'\| = \|A\|\|A'\| = (-3)(-3) = 9$
3.	(b)	The area of the parallelogram with adjacent sides AB and AC = $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$. Hence, the area of the triangle with vertices A, B, C $=\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$
4.	(c)	The function f is continuous at x = 0 if $lim_{x\to 0}f(x) = f(0)$ We have f(0) = k and $lim_{x\to 0}f(x) = lim_{x\to 0}\frac{1-cos}{8x^2} = lim_{x\to 0}\frac{2sin^2 2x}{8x^2} = lim_{x\to 0}\frac{sin^2 2x}{4x^2}$ $= lim_{x\to 0}\left(\frac{sin2x}{2x}\right)^2 = 1$ Hence, k =1
5.	(b)	$\frac{x^2}{2} + \log\ \|x\| + C\left(\because f(x) = \int\left(x + \frac{1}{x}\right)dx\right)$
6.	(c)	The given differential equation is $4\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2} = 0$. Here, m = 2 and n = 1 Hence, m + n= 3
7.	(b)	The strict inequality represents an open half plane and it contains the origin as (0, 0) satisfies it.
8.	(a)	Scalar Projection of $3\hat{\imath} - \hat{\jmath} - 2\hat{k}\ on\ vector\ \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ $=\frac{(3\hat{\imath}-\hat{\jmath}-2\hat{k}).(\hat{\imath}+2\hat{\jmath}-3\hat{k})}{\|\hat{\imath}+2\hat{\jmath}-3\hat{k}\|} = \frac{7}{\sqrt{14}}$
9.	(c)	$\int_2^3\frac{x}{x^2+1} = \frac{1}{2}[log(x^2+1)]_2^3 = \frac{1}{2}(log10 - log5) = \frac{1}{2}log\left(\frac{10}{5}\right)$ $= \frac{1}{2}log2$
10.	(c)	$(AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = BA^{-1}$
11.	(d)	The minimum value of the objective function occurs at two adjacent corner points (0.6, 1.6) and (3, 0) and there is no point in the half plane $4x + 6y < 12$ in common with the feasible region. So, the minimum value occurs at every point of the line-segment joining the two points.
12.	(d)	$2 - 20 = 2x^2 - 24 \Rightarrow 2x^2 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$
13.	(b)	$\|adjA\| = \|A\|^{n-1} \Rightarrow \|adjA\| = 25$
14.	(c)	P$(A' \cap B') = P(A') \times P(B')$ (As A and B are independent, $A'\ and\ B'$ are also independent.) = 0.7 × 0.4 = 0.28

15.	(c)	$ydx - xdy = 0 \Rightarrow ydx - xdy = 0 \Rightarrow \frac{dy}{y} = \frac{dx}{x}$ $\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} + logK, K > 0 \Rightarrow \log\|y\| = \log\|x\| + \log K$ $\Rightarrow \log\|y\| = \log\ \|x\|K \Rightarrow \|y\| = \|x\|K \Rightarrow$ y=$\pm Kx \Rightarrow y = Cx$
16.	(a)	$y = \sin^{-1}x$ $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,.\frac{dy}{dx} = 1$ Again , differentiating both sides w. r. to x, we get $\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}.\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = 0$ Simplifying, we get $(1-x^2)y_2 = xy_1$
17.	(b)	$\left\|\vec{a} - 2\vec{b}\right\|^2 = (\vec{a} - 2\vec{b}).(\vec{a} - 2\vec{b})$ $\left\|\vec{a} - 2\vec{b}\right\|^2 = \vec{a}.\vec{a} - 4\vec{a}.\vec{b} + 4\vec{b}.\vec{b}$ $= \|\vec{a}\|^2 - 4\vec{a}.\vec{b} + 4\left\|\vec{b}\right\|^2$ $= 4 - 16 + 36 = 24$ $\left\|\vec{a} - 2\vec{b}\right\|^2 = 24 \Rightarrow \left\|\vec{a} - 2\vec{b}\right\| = 2\sqrt{6}$
18.	(b)	The line through the points (0, 5, -2) and (3, -1, 2) is $\frac{x}{3-0} = \frac{y-5}{-1-5} = \frac{z+2}{2+2}$ $or, \frac{x}{3} = \frac{y-5}{-6} = \frac{z+2}{4}$ Any point on the line is $(3k, -6k+5, 4k-2)$, where k is an arbitrary scalar. $3k = 6 \Rightarrow k = 2$ The z-coordinate of the point P will be $4 \times 2 - 2 = 6$
19.	(c)	$sec^{-1}x$ is defined if $x \le -1$ or $x \ge 1$. Hence, $sec^{-1}2x$ will be defined if $x \le -\frac{1}{2}$ or $x \ge \frac{1}{2}$. Hence, A is true. The range of the function $sec^{-1}x$ is $[0, \pi] - \{\frac{\pi}{2}\}$ R is false.
20.	(a)	The equation of the x-axis may be written as $\vec{r} = t\hat{\imath}$. Hence, the acute angle θ between the given line and the x-axis is given by $cos\theta = \frac{\|1 \times 1 + (-1) \times 0 + 0 \times 0\|}{\sqrt{1^2 + (-1)^2 + 0^2} \times \sqrt{1^2 + 0^2 + 0^2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

SECTION B (VSA questions of 2 marks each)

21.	$sin^{-1}[sin\left(\frac{13\pi}{7}\right)] = sin^{-1}[sin\left(2\pi - \frac{\pi}{7}\right)]$ $= sin^{-1}[sin\left(-\frac{\pi}{7}\right)] = -\frac{\pi}{7}$ **OR**	1 1

	Let $y \in N$(codomain). Then $\exists\ 2y \in N$(domain) such that $f(2y) = \frac{2y}{2} = y$. Hence, f is surjective. $1, 2 \in N$(domain) such that $f(1) = 1 = f(2)$ Hence, f is not injective.	1 1
22.	Let AB represent the height of the street light from the ground. At any time t seconds, let the man represented as ED of height 1.6 m be at a distance of x m from AB and the length of his shadow EC be y m. Using similarity of triangles, we have $\frac{4}{1.6} = \frac{x+y}{y} \Rightarrow 3y = 2x$  Differentiating both sides w.r.to t, we get $3\frac{dy}{dt} = 2\frac{dx}{dt}$ $$\frac{dy}{dt} = \frac{2}{3} \times 0.3 \Rightarrow \frac{dy}{dt} = 0.2$$ At any time t seconds, the tip of his shadow is at a distance of $(x+y)$ m from AB. The rate at which the tip of his shadow moving $$= \left(\frac{dx}{dt} + \frac{dy}{dt}\right) m/s = 0.5\ m/s$$ The rate at which his shadow is lengthening $$= \frac{dy}{dt}\ m/s = 0.2\ m/s$$	½ ½ ½ 1/2
23.	$\vec{a} = \hat{\imath} - \hat{\jmath} + 7\hat{k}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath} + \lambda\hat{k}$ Hence $\vec{a} + \vec{b} = 6\hat{\imath} - 2\hat{\jmath} + (7+\lambda)\hat{k}$ and $\vec{a} - \vec{b} = -4\hat{\imath} + (7-\lambda)\hat{k}$ $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ will be orthogonal if, $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$ i.e., if, $-24 + (49 - \lambda^2) = 0 \Longrightarrow \lambda^2 = 25$ i.e., if, $\lambda = \pm 5$ **OR** The equations of the line are $6x - 12 = 3y + 9 = 2z - 2$, which, when written in standard symmetric form, will be $\frac{x-2}{\frac{1}{6}} = \frac{y-(-3)}{\frac{1}{3}} = \frac{z-1}{\frac{1}{2}}$ Since, lines are parallel, we have $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ Hence, the required direction ratios are $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$ or (1,2,3)	½ ½ 1 ½ ½

	and the required direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$	1
24.	$y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ Let $sin^{-1}x = A$ and $sin^{-1}y = B$. Then x = sinA and y = sinB $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1 \Rightarrow sinBcosA + sinAcosB = 1$ $\Rightarrow \sin(A+B) = 1 \Rightarrow A + B = sin^{-1}1 = \frac{\pi}{2}$ $\Rightarrow\ sin^{-1}x + sin^{-1}y = \frac{\pi}{2}$ Differentiating w.r.to x, we obtain $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$	½ ½ 1
25.	Since $\vec{\boldsymbol{a}}$ is a unit vector, $\therefore\ \|\vec{a}\| = 1$ $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12.$ $\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 12$ $\Rightarrow \|\vec{x}\|^2 - \|\vec{a}\|^2 = 12.$ $\Rightarrow \|\vec{x}\|^2 - 1 = 12$ $\Rightarrow \|\vec{x}\|^2 = 13 \Rightarrow \|\vec{x}\| = \sqrt{13}$	½ ½ ½ 1/2

SECTION C

(*Short Answer Questions of 3 Marks each*)

26.	$\int \frac{dx}{\sqrt{3 - 2x - x^2}}$ $= \int \frac{dx}{\sqrt{-(x^2+2x-3)}} = \int \frac{dx}{\sqrt{4-(x+1)^2}}$ $= sin^{-1}\left(\frac{x+1}{2}\right) + C\ \ [\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right) + C\ ]$	 2 1
27.	P(not obtaining an odd person in a single round) = P(All three of them throw tails or All three of them throw heads) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 2 = \frac{1}{4}$ P(obtaining an odd person in a single round) = 1 − P(not obtaining an odd person in a single round) = $\frac{3}{4}$ The required probability = P('In first round there is no odd person' and 'In second round there is no odd person' and 'In third round there is an odd person') $= \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{64}$	 1+1/2 ½ 1

OR

Let X denote the Random Variable defined by the number of defective items.

P(X=0) $= \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$

P(X=1) = $2 \times \left(\frac{2}{6} \times \frac{4}{5}\right) = \frac{8}{15}$

P(X=2) = $\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$ — 2

x_i	0	1	2
p_i	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$
$p_i x_i$	0	$\frac{8}{15}$	$\frac{2}{15}$

1/2

Mean = $\sum p_i x_i = \frac{10}{15} = \frac{2}{3}$ — 1/2

28.

Let I = $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tanx}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}} dx$ ..(i)

Using $\int_a^b \boldsymbol{f(x)\ dx} = \int_a^b \boldsymbol{f(a+b-x)dx}$

I = $\int_{\pi/6}^{\pi/3} \frac{\sqrt{cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}} dx$

I = $\int_{\pi/6}^{\pi/3} \frac{\sqrt{sinx}}{\sqrt{cosx}+\sqrt{sinx}}$ dx ..(ii). — 1

Adding (i) and (ii), we get

2I = $\int_{\pi/6}^{\pi/3} \frac{\sqrt{cosx}}{\sqrt{sinx}+\sqrt{cosx}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{sinx}}{\sqrt{cosx}+\sqrt{sinx}}$dx

2I = $\int_{\pi/6}^{\pi/3} dx$ — 1

$= [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

Hence, I = $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tanx}} = \frac{\pi}{12}$ — 1

OR

$\int_0^4 |x-1|\, dx = \int_0^1 (1-x)dx + \int_1^4 (x-1)dx$ — 1

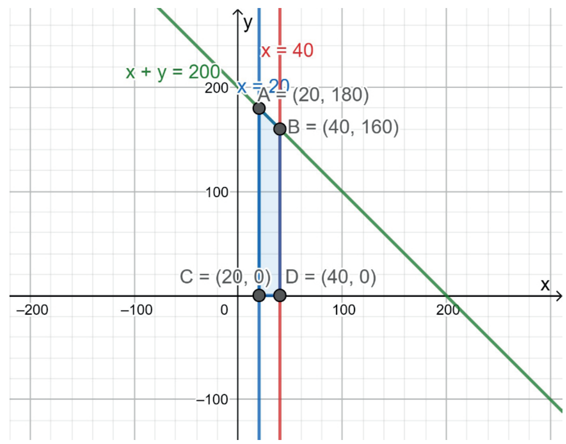
	$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$ $=(1-\frac{1}{2})+(8-4)-(\frac{1}{2}-1)$ $=5$	1 1
29.	$ydx+(x-y^2)dy=0$ Reducing the given differential equation to the form $\frac{dx}{dy}+\boldsymbol{Px}=\boldsymbol{Q}$ we get, $\frac{dx}{dy}+\frac{x}{y}=y$ I.F = $e^{\int Pdy}=e^{\int\frac{1}{y}dy}=e^{logy}=y$ The general solution is given by $x.IF=\int Q.IFdy \Rightarrow xy=\int y^2dy$ $\Rightarrow xy=\frac{y^3}{3}+C$, which is the required general solution **OR** $xdy-ydx=\sqrt{x^2+y^2}\,dx$ It is a **Homogeneous Equation as** $\frac{dy}{dx}=\frac{\sqrt{x^2+y^2}+y}{x}=\sqrt{1+(\frac{y}{x})^2}+\frac{y}{x}=f\left(\frac{y}{x}\right).$ Put $y=vx$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$ $v+x\frac{dv}{dx}=\sqrt{1+v^2}+v$ Separating variables, we get $\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$ Integrating, we get $log\left\|v+\sqrt{1+v^2}\right\|=log\|x\|+logK, K>0$ $log\left\|y+\sqrt{x^2+y^2}\right\|=logx^2K$ $\Rightarrow y+\sqrt{x^2+y^2}=\pm Kx^2$ $\Rightarrow y+\sqrt{x^2+y^2}=Cx^2$, which is the required general solution	 ½ 1 1 ½ ½ ½ ½ 1+1/2

30.

We have Z= 400x +300y subject to

$x + y \leq 200, x \leq 40, x \geq 20, y \geq 0$

The corner points of the feasible region are C(20,0), D(40,0), B(40,160), A(20,180)



1

Corner Point	Z = 400x + 300y
C(20,0)	8000
D(40,0)	16000
B(40,160)	64000
A(20,180)	62000

1

Maximum profit occurs at x= 40, y=160

and the maximum profit =₹ 64,000 1

31.

$\int \frac{(x^3+x+1)}{(x^2-1)} dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)}\right) dx$ 1

Now resolving $\frac{2x+1}{(x-1)(x+1)}$ into partial fractions as

$$\frac{2x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

We get $\frac{2x+1}{(x-1)(x+1)} = \frac{3}{2(x-1)} + \frac{1}{2(x+1)}$ 1

Hence, $\int \frac{(x^3+x+1)}{(x^2-1)} dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)}\right) dx$

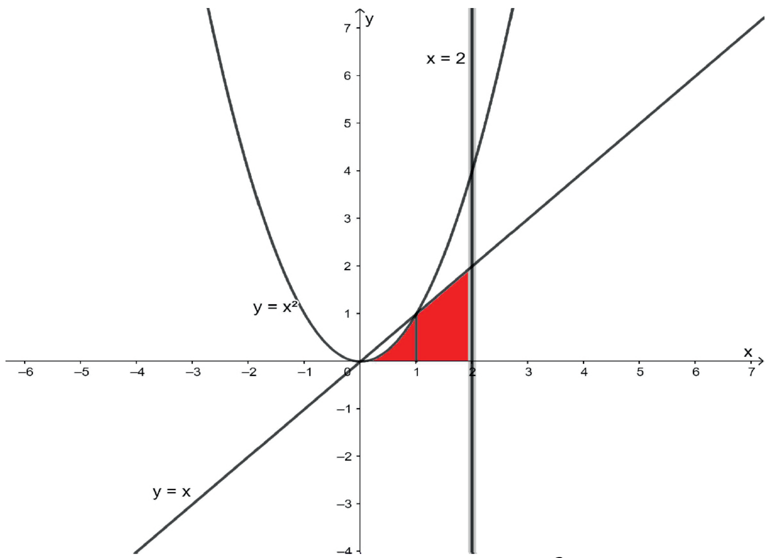
$= \int \left(x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}\right) dx$

$= \frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$

$= \frac{x^2}{2} + \frac{1}{2}(\log|(x-1)^3(x+1)| + C$ 1

SECTION D

(Long answer type questions (LA) of 5 marks each)

32.	 The points of intersection of the parabola $y = x^2$ and the line $y = x$ are (0, 0) and (1, 1). Required Area = $\int_0^1 y_{parabola}\, dx + \int_1^2 y_{line}\, dx$ Required Area = $\int_0^1 x^2\, dx + \int_1^2 x\, dx$ $= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}$	(Correct Fig: 1 Mark) ½ 2 1+1/2
33.	Let $(a, b) \in N \times N$. Then we have ab = ba (by commutative property of multiplication of natural numbers) $\Rightarrow (a, b)R(a, b)$ Hence, R is reflexive. Let $(a, b), (c, d) \in N \times N$ such that (a, b) R (c, d). Then ad = bc $\Rightarrow cb = da$ (by commutative property of multiplication of natural numbers $\Rightarrow (c, d)R(a, b)$ Hence, R is symmetric. Let $(a, b), (c, d), (e, f) \in N \times N$ such that (a, b) R (c, d) and (c, d) R (e, f). Then ad = bc, cf = de $\Rightarrow adcf = bcde$ $\Rightarrow af = be$ $\Rightarrow (a, b)R(e, f)$	1 1+1/2

	Hence, R is transitive.	2
	Since, R is reflexive, symmetric and transitive, R is an	
	equivalence relation on $N \times N$.	½
	OR	
	Let $A \in P(X)$. Then $A \subset A$	
	$\Longrightarrow (A, A) \in R$	
	Hence, R is reflexive.	1
	Let $A, B, C \in P(X)$ such that	
	$(A, B), (B, C) \in R$	
	$\Longrightarrow A \subset B, B \subset C$	
	$\Longrightarrow A \subset C$	
	$\Longrightarrow (A, C) \in R$	
	Hence, R is transitive.	2
	$\emptyset, X \in P(X)$ such that $\emptyset \subset X$. Hence, $(\emptyset, X) \in R$. But, $X \not\subset \emptyset$,	
	which implies that $(X, \emptyset) \notin R$.	
	Thus, R is not symmetric.	2
34.	The given lines are non-parallel lines. There is a unique line-segment PQ (P lying on one and Q on the other, which is at right angles to both the lines. PQ is the shortest distance between the lines. Hence, the shortest possible distance between the insects = PQ	
	The position vector of P lying on the line	
	$\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \lambda(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$	
	is $(6 + \lambda)\hat{\imath} + (2 - 2\lambda)\hat{\jmath} + (2 + 2\lambda)\hat{k}$ for some λ	½
	The position vector of Q lying on the line	
	$\vec{r} = -4\hat{\imath} - \hat{k} + \mu(3\hat{\imath} - 2\hat{\jmath} - 2\hat{k})$	
	is $(-4 + 3\mu)\hat{\imath} + (-2\mu)\hat{\jmath} + (-1 - 2\mu)\hat{k}$ for some μ	½
	$\overrightarrow{PQ} = (-10 + 3\mu - \lambda)\hat{\imath} + (-2\mu - 2 + 2\lambda)\hat{\jmath} + (-3 - 2\mu - 2\lambda)\hat{k}$	½
	Since, PQ is perpendicular to both the lines	
	$(-10 + 3\mu - \lambda) + (-2\mu - 2 + 2\lambda)(-2) + (-3 - 2\mu - 2\lambda)2 = 0,$	½
	$i.e., \mu - 3\lambda = 4$ …(i)	
	and $(-10 + 3\mu - \lambda)3 + (-2\mu - 2 + 2\lambda)(-2) + (-3 - 2\mu - 2\lambda)(-2) = 0,$	½
	$i.e., 17\mu - 3\lambda = 20$ …(ii)	
	solving (i) and (ii) for λ and μ, we get $\mu = 1, \lambda = -1$.	1
	The position vector of the points, at which they should be so that the distance between them is the shortest, are	
	$5\hat{\imath} + 4\hat{\jmath}$ and $-\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$	½
	$\overrightarrow{PQ} = -6\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$	
	The shortest distance = $\lvert\overrightarrow{PQ}\rvert = \sqrt{6^2 + 6^2 + 3^2} = 9$	1
	OR	

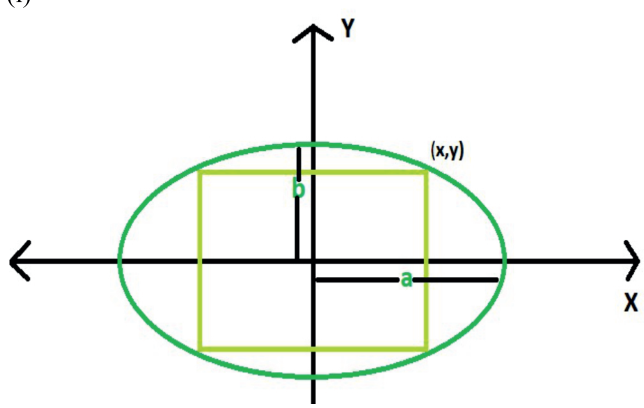
	Eliminating t between the equations, we obtain the equation of the path $\frac{x}{2}=\frac{y}{-4}=\frac{z}{4}$, which are the equations of the line passing through the origin having direction ratios <2, -4, 4>. This line is the path of the rocket. When t = 10 seconds, the rocket will be at the point (20, -40, 40). Hence, the required distance from the origin at 10 seconds = $\sqrt{20^2+40^2+40^2}km = 20\times 3\ km = 60\ km$ The distance of the point (20, -40, 40) from the given line $=\frac{\lvert(\vec{a_2}-\vec{a_1})\times\vec{b}\rvert}{\lvert\vec{b}\rvert} = \frac{\lvert -30\hat{j}\times(10\hat{i}-20\hat{j}+10\hat{k})\rvert}{\lvert 10\hat{i}-20\hat{}+10\hat{k}\rvert}\ km = \frac{\lvert -300\hat{i}+300\hat{k}\rvert}{\lvert 10\hat{i}-20\hat{j}+10\hat{k}\rvert}\ km$ $=\frac{300\sqrt{2}}{10\sqrt{6}}\ km = 10\sqrt{3}\ km$	1 ½ 1 2 1/2
35.	$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$ $\lvert A\rvert = 2(0)+3(-2)+5(1) = -1$ $A^{-1}=\frac{adjA}{\lvert A\rvert}$ $adjA=\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}, A^{-1}=\frac{1}{(-1)}\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$ $X = A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{(-1)}\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$ $=\frac{1}{(-1)}\begin{bmatrix}0+5-6\\ 22+45-69\\ 11+25-39\end{bmatrix}$ $\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{(-1)}\begin{bmatrix}-1\\ -2\\ -3\end{bmatrix} \Rightarrow x=1, y=2, z=3.$	½ 3 1+1/2

SECTION E(Case Studies/Passage based questions of 4 Marks each)

36.	(i) $f(x) = -0.1x^2 + mx + 98.6$, being a polynomial function, is differentiable everywhere, hence, differentiable in (0, 12)	1
	(ii) $f'(x) = -0.2x + m$ Since, 6 is the critical point, $f'(6) = 0 \Rightarrow m = 1.2$	1
	(iii) $f(x) = -0.1x^2 + 1.2x + 98.6$ $f'(x) = -0.2x + 1.2 = -0.2(x - 6)$	
	(see table below)	1+1
	OR (iii) $f(x) = -0.1x^2 + 1.2x + 98.6,$ $f'(x) = -0.2x + 1.2, f'(6) = 0,$ $f''(x) = -0.2$ $f''(6) = -0.2 < 0$ Hence, by second derivative test 6 is a point of local maximum. The local maximum value = $f(6) = -0.1 \times 6^2 + 1.2 \times 6 + 98.6 = 102.2$	1
	We have $f(0) = 98.6, f(6) = 102.2, f(12) = 98.6$ 6 is the point of absolute maximum and the absolute maximum value of the function = 102.2.	1/2
	0 and 12 both are the points of absolute minimum and the absolute minimum value of the function = 98.6.	1/2
37.	(i)	

Table for 36 (iii):

In the Interval	$f'(x)$	Conclusion
(0, 6)	+ve	f is strictly increasing in [0, 6]
(6, 12)	-ve	f is strictly decreasing in [6, 12]



Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle.

The area function $A = 2x \times 2\frac{b}{a}\sqrt{a^2 - x^2}$

	$=\frac{4b}{a}x\sqrt{a^2-x^2}, x\in(0,a)$. (ii)$\frac{dA}{dx}=\frac{4b}{a}\left[x\times\frac{-x}{\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}\right]$ $=\frac{4b}{a}\times\frac{a^2-2x^2}{\sqrt{a^2-x^2}}=-\frac{4b}{a}\times\frac{2\left(x+\frac{a}{\sqrt{2}}\right)(x-\frac{a}{\sqrt{2}})}{\sqrt{a^2-x^2}}$ $\frac{dA}{dx}=0\Rightarrow x=\frac{a}{\sqrt{2}}$. $x=\frac{a}{\sqrt{2}}$ is the critical point. (iii)For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}>0$ and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}<0$. Hence, by the first derivative test, there is a local maximum at the critical point $x=\frac{a}{\sqrt{2}}$. Since there is only one critical point, therefore, the area of the soccer field is maximum at this critical point $x=\frac{a}{\sqrt{2}}$ Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$. OR (iii) $A=2x\times2\frac{b}{a}\sqrt{a^2-x^2}, x\in(0,a)$. Squaring both sides, we get $Z=A^2=\frac{16b^2}{a^2}x^2(a^2-x^2)=\frac{16b^2}{a^2}(x^2a^2-x^4), x\in(0,a)$. A is maximum when Z is maximum. $\frac{dZ}{dx}=\frac{16b^2}{a^2}(2xa^2-4x^3)=\frac{32b^2}{a^2}x(a+\sqrt{2}x)(a-\sqrt{2}x)$ $\frac{dZ}{dx}=0\Rightarrow x=\frac{a}{\sqrt{2}}$. $\frac{d^2Z}{dx^2}=\frac{32b^2}{a^2}(a^2-6x^2)$ $(\frac{d^2Z}{dx^2})_{x=\frac{a}{\sqrt{2}}}=\frac{32b^2}{a^2}(a^2-3a^2)=-64b^2<0$ Hence, by the second derivative test, there is a local maximum value of Z at the critical point $x=\frac{a}{\sqrt{2}}$. Since there is only one critical point, therefore, Z is maximum at $x=\frac{a}{\sqrt{2}}$, hence, A is maximum at $x=\frac{a}{\sqrt{2}}$. Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.	1 ½ 1/2 1 1/2 ½ 1 1/2 1/2
38.	(i)Let P be the event that the shell fired from A hits the plane and Q be the event that the shell fired from B hits the plane. The following four hypotheses are possible before the trial, with the guns operating independently: $E_1=PQ, E_2=\bar{P}\bar{Q}, E_3=\bar{P}Q, E_4=P\bar{Q}$ Let E = The shell fired from exactly one of them hits the plane. $P(E_1)=0.3\times0.2=0.06, P(E_2)=0.7\times0.8=0.56, P(E_3)=0.7\times0.2$ $=0.14, P(E_4)=0.3\times0.8=0.24$ $P\left(\frac{E}{E_1}\right)=0, P\left(\frac{E}{E_2}\right)=0, P\left(\frac{E}{E_3}\right)=1, P\left(\frac{E}{E_4}\right)=1$	1

	$\left(E_1\right)$ $\quad$ $\left(E_2\right)$ $\quad$ $\left(E_3\right)$ $\quad$ $\left(E_4\right)$	1
	$P(E) = P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2).P\left(\frac{E}{E_2}\right) + P(E_3).P\left(\frac{E}{E_3}\right) + P(E_4).P\left(\frac{E}{E_4}\right)$	
	= 0.14 + 0.24 = 0.38	1
	(ii)By Bayes' Theorem, $P\left(\frac{E_3}{E}\right) = \frac{P(E_3).P\left(\frac{E}{E_3}\right)}{P(E_1).P\left(\frac{E}{E_1}\right)+P(E_2).P\left(\frac{E}{E_2}\right)+P(E_3).P\left(\frac{E}{E_3}\right)+P(E_4).P\left(\frac{E}{E_4}\right)}$	
	$= \frac{0.14}{0.38} = \frac{7}{19}$	2
	NOTE: The four hypotheses form the partition of the sample space and it can be seen that the sum of their probabilities is 1. The hypotheses $E_1 and\ E_2$ are actually eliminated as $P\left(\frac{E}{E_1}\right) = P\left(\frac{E}{E_2}\right) = 0$	
	Alternative way of writing the solution:	
	(i)P(Shell fired from exactly one of them hits the plane)	
	= P[(Shell from A hits the plane and Shell from B does not hit the plane) or (Shell from A does not hit the plane and Shell from B hits the plane)]	1
	$= 0.3 \times 0.8 + 0.7 \times 0.2 = 0.38$	1
	(ii)P(Shell fired from B hit the plane/Exactly one of them hit the plane)	
	$= \frac{\text{P(Shell fired from B hit the plane} \cap \text{Exactly one of them hit the plane)}}{\text{P(Exactly one of them hit the plane)}}$	
	$= \frac{P(Shell\ from\ only\ B\ hit\ the\ plane)}{P(Exactly\ one\ of\ them\ hit\ the\ plane)}$	1
	$= \frac{0.14}{0.38} = \frac{7}{19}$	1

Directorate of Education, GNCT of Delhi

(PRACTICE PAPER-I)

(2022-23)

Class – XII

Mathematics (Code: 041)

Time: 3 hours **Maximum Marks: 80**

General Instructions :

1. This Question paper contains - F**ive Sections A,B,C,D,E**. Each section is compulsory. However, there are internal choices in some questions.
2. **Section A** has 18 **MCQ's and 02** Assertion-Reason based questions of 1 mark each.(20 Marks)
3. **Section B**has 5 **Very Short Answer (VSA)-type** questions of 2 marks each.(10 Marks)
4. **Section C** has 6 **Short Answer (SA)-type** questions of 3 marks each.(18 Marks)
5. **Section D** has 4 **Long Answer (LA)-type** questions of 5 marks each.(20 Marks)
6. **Section E has 3 Source based/Case based/passage based/integrated units of assessment (4 marks each) with sub parts.(12 Marks)**

	Section – A	
Q .NO	Question Number 1-18 are of MCQ type question one mark each.	Marks
1	If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of x+y is : (a) x=3 , y=1 (b) x=2 , y=3 (c) x=2 , y=4 (d) x=3 , y=3	**1**
2	If $A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $n \in N$, then a^n equals to : (a) na (b) $2na$ (c) $2^{n-1}a$ (d) $2^n a$	**1**
3	If the matrix $\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is a skew symmetric , then (a, b) is : (a) 1, -3 (b)-1, 1 (c) -2, 3 (d) 0, 0	**1**

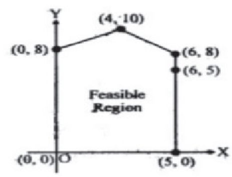
4 .	If A and B are square matrices of order 3 such that $\|A\|=5$ and ab=-5 I , Then value of $\|B\|$ is :		1
	(a) -5	(b) -25	
	(c) 25	(d) इनमें से कोई नहीं/None of these	
5	For what value of k inverse does not exists for matrix $\begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix}$?		
	(a) 0	(b)3	
	(c) 6	(d) 2	
6	If t $f(x)=\begin{cases} ax^2+1, & x>1 \\ x+a & x\leq 1 \end{cases}$ *is derivable at* $x=1$ then the value of a is :		
	(a) 0	(b) 1	
	(c) 1/2	(d) 2	
7.	*Derivative of* $\cos^{-1}(2x^2-1)$ $w.r.t$ $\cos^{-1}x$ is:		
	(a) 2	(b) $\frac{2}{x}$	
	(c) $1-x^2$	(d) $\frac{-1}{2\sqrt{1-x^2}}$	
8	$\int x^2 e^{x^3} dx$ is given by :		1
	(a) $\frac{1}{3}e^{x^3}+C$	(b) $\frac{1}{3}e^{x^2}+C$	
	(c) $\frac{1}{2}e^{x^3}+C$	(d) $\frac{1}{2}e^{x^2}+C$	
9	*If* $f(x)=\begin{cases} 2x+8, & if\ 1\leq x\leq 2 \\ 6x & if\ 2<x\leq 4 \end{cases}$, then $\int_1^4 f(x)dx$ is :		
	(a) 43	(b)44	
	(c) 47	(d) 46	

10	The order and degree of the differential equation $\left[\frac{d^3y}{dx^3}\right]^2-3\frac{d^2y}{dx^2}+2\left[\frac{dy}{dx}\right]^4=y^4$ are		
	(a) $1,4$	(b) $3,4$	
	(c) $2,4$	(d) $3,2$	
11	The integrating factor of the differential Equation $\frac{dy}{dx}+y=\frac{1+y}{x}$ is :		**1**
	(a) $\frac{x}{e^x}$	(b) $\frac{e^x}{x}$	
	(c) xe^x	(d) e^x	
12	The projection of the vector $2\hat{i}-\hat{j}+\hat{k}$ on the vector $\hat{i}-2\hat{j}+\hat{k}$ is :		
	(a) $\frac{4}{\sqrt{6}}$	(b) $\frac{5}{\sqrt{6}}$	
	(c) $\frac{4}{\sqrt{3}}$	(d) $\frac{7}{\sqrt{6}}$	
13	**Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is angle between them . Then $\vec{a}+\vec{b}$ is unit vector if θ is equal to :**		
	(a) $\frac{\pi}{4}$	(b) $\frac{\pi}{3}$	
	(c) $\frac{\pi}{2}$	(d) $\frac{2\pi}{3}$	
14	If $\lvert\vec{a}\times\vec{b}\rvert=\lvert\vec{a}.\vec{b}\rvert$ then the angle between $\vec{a}$ and $\vec{b}$ is equals to:		**1**
	(a) 0	(b) $\frac{\pi}{2}$	
	(c) $\frac{\pi}{4}$	(d) π	

15 The reflection of the point (α, β, γ) in the xy- plane is : **1**

(a) $(\alpha, \beta, 0)$	(b) $(0, 0, \gamma)$
(c) $(-\alpha, -\beta, \gamma)$	(d) (α, β, γ)

16 The feasible region for a LPP is shown below. Let Z=3x-4y be the objective function. Minimum of Z occurs at



(a) $(0, 0)$	(b) $(0, 8)$
(c) $(5, 0)$	(d)

17 If A , B are two events associated with same random experiment such that P(A)=0.4, P(B)=0.8 and $P(\frac{B}{A})$ =0.6 then $P(\frac{A}{B})$ is :

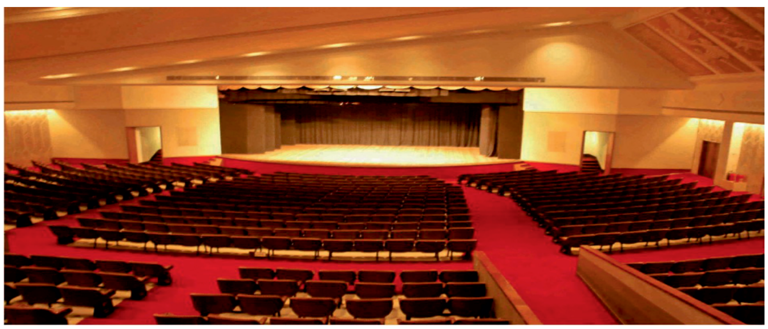
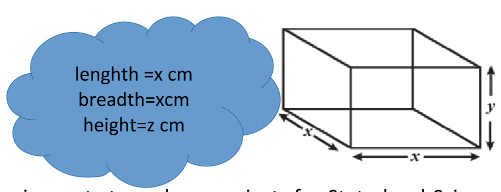
(a) 0.3	(b) 0.4
(c) 0.5	(d) 0.6

18 The corner points of a feasible region of a LPP are (0,2), (3,0), (6, 0), (6, 8) and (0, 5). The minimum value of Z=4x+6y occurs:

(a) (0, 2) (only)	(b) (3, 0) (only)
(c) The mid point of the line segment joining the points (0, 2) and (3, 0) only	(d) Any point on the line segment j oining the points (0,2) and (3,0)

	(ASSERTION-REASON BASED QUESTIONS) **In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.** **(a) Both A and R are true and R is the correct explanation of A.** **(b) Both A and R are true but R is not the correct explanation of A.** **(c) A is true but R is false.** **(d) A is false but R is true.**	
19	**Assertion :(A)** **The domain of the function** $\sin^{-1}(2x-1)$ **is [0, 1]** **Reason (R):The domain of the function** $\sin^{-1}x$ **is [-1, 1]**	
20	**Assertion(A):The position of a particle in a rectangular coordinate system is (3, 2, 5) then its position vector will be** $2\hat{i}+5\hat{j}+3\hat{k}$ **Reason (R): Displacement vector of the particle that moves from point P(2 ,3,5) to point Q(3, 4, 5) is** $\hat{i}+\hat{j}$	
	<u>Section B</u> **This section contains 5 Very Short Answer (VSA)-type questions of 2 marks each.**	
21.	Let $f: N \to N$ be defined by $f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n \in N$ Find whether function is bijective . Justify your answer . **OR** Find the value of the following: $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$	2
22.	A man of height 2m walks at uniform speed of 5km/hr away from a lamp post which is 6m high. Find the rate of which the length of his shadow increases.	2
23.	Find the unit vector perpendicular to each of the vectors $\lvert\vec{a}\rvert=4\hat{i}+3\hat{j}+\hat{k}$ and $\lvert\vec{b}\rvert=2\hat{i}-\hat{j}+2\hat{k}$ **OR** If the line through the points (4 ,1 ,2) and (5 λ , 0) is parallel to the line through the points (2 ,1 ,1) and (3 ,3 ,-1) , find λ	2

24.	If $If\ x\sqrt{1+y}+y\sqrt{1+x}=0\ for -1<x<1$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$	**2**
25.	If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})(\vec{x}+\vec{a})=15$ then find $\lvert\vec{x}\rvert$.	**2**
	Section C **This Section Contains 6 Short Answer (SA)-Type Questions of 3 Marks Each.**	
26.	Find $\int \frac{dx}{\sqrt{(x-a)(x-b)}}$	**3**
27.	Bag I contains 3 red and 4 black balls and bag II contains 4 red and 5 black balls . One ball is transferred from bag I to bag II without seeing its colour . A ball is then drawn from bag II . If the drawn ball is red in colour find the probability that transferred ball is black. **OR** Three cards are drawn at random (without replacement)from a well shuffled pack of 52 playing cards . Fin the probability distribution of number of red cards. Hence find the mean of distribution.	**3**
28 .	Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{sinx}+\sqrt{\cos x}}dx$ OR $\int_2^8 \lvert x-5\rvert dx$	**3**
29.	Solve the differential equation $ydx-(x+2y^2)dy=0$ **OR** Solve the differential equation $(x-y)dy-(x+y)dx=0$	**3**
30	Solve the following LPP graphically; Minimize Z=5x+10y Subj ect to constraints $x+2y\leq 120, x+y\geq 60, x-2y\geq 0, x, y\geq 0$	**3**
31.	Find $\int \frac{1}{x(x^4-1)}dx$	**3**
	<u>Section D</u> This section contains **four Long Answer (LA)-type** questions of 5marks each.	
32.	Make a sketch of region $\{(x,y): 0\leq y\leq x^2+3, 0\leq y\leq 2x+3, 0\leq x\leq 3\}$ and find its area using integration .	**5**
33.	Show that the relation R in the set A ={1, 2, 3, 4, 5} given by R={(a, b):|a-b| is divisible by 2} is an equivalence relation. **OR** Given A={2, 3, 4} , B= {2, 5, 6, 7). Construct an example of each of following (a)an inj ective mapping from A to B (b) A mapping from A to B which is nit inj ective (c)A mapping from B to A	**5**

34.	Find the shortest distance between the lines given by $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$ **OR** Find the foot of perpendicular from the point (2, 3, -8) to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$ Also find the perpendicular diastance from the given point to the line.	
35.	If A=$\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$, then find the value of A^{-1}. Using A^{-1} solve the system of linear equations 2x -3y+5z =11 , 3x +2y - 4z=-5 and x +y -2z=-3 .	**5**
	(Section E) **Source based/Case based/passage based/integrated units of assessment Questions**	
36.	 A architect design a auditorium for a school for its cultural activities .The floor of the auditorium is rectangular in shapeand has a fixed parameter P. Based on the above information answer the following questions. (i)Express area of a rectangular region as a function of x . (ii)School manager is intrested in maximising the area of floor 'A'for this to be happen what should be the value of x? (iii)Find the value of y for the area of floor to be maximum. OR What will be maximum area of the floor?	
37.	 Anuja wants to make a project for State level Science Exhibition . For this she wants to make metal box with Square base and verticle sides to contain of $1024\,cm^3$ water material for top and bottom costs ₹ $5\,per\,cm^2$ and material for slides costs ₹ $2.5\,per\,cm^2$	

	Based on the above information answer the following: (i) What will be relation between x and y? (ii) What will be the total cost (C) of the material used to construct the box ? (iii) What will be the total cost (C) of the box in terms of x? **OR** (iii)What is the least cost of the box ?	**1** **1** **2**
38.	 A shopkeeper sells three type of flower seeds A_1, A_2, A_3 . They are sold in the form of mixture, where proportion of these seeds are 4: 4 :2, respectively. Germination rates of the three type of seeds are 45%, 60% and 35% respectively. Based on the above information : (i)Calculate the probability that a randomly choosen seed will germinate. (ii)Calculate the probability that the seed is of the type A_2 , given that randomly chosen seed germinates.	**2+2**

Directorate of Education, GNCT of Delhi

(PRACTICE PAPER -I)

(2022-23)

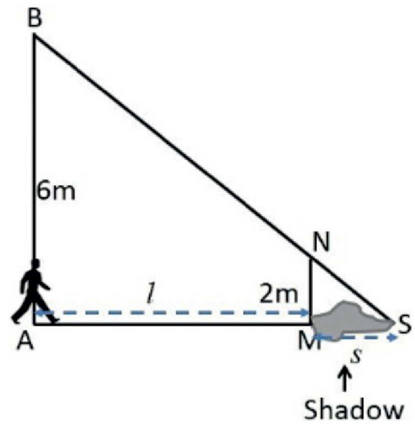
Class – XII

Mathematics (Code: 041)

MARKING SCHEME

	Section – A	
1	(b) x=2 , y=3	1
2	(c) $2^{n-1}a$	1
3	(c) -2, 3	1
4	(b)-25	1
5	(b)3	1
6	(c) 1/2	1
7.	(a) 2	1
8	(a) $\frac{1}{3}e^{x^3}+C$	1
9	(c) 47	1
10	(d) 3 ,2	1
11	(b) $\frac{e^x}{x}$	1
12	(b) $\frac{5}{\sqrt{6}}$	1
13	**(d)**	1

14	(a) 0	**1**
15	(d) (α, β, γ) :	**1**
16	(b) $(0,8)$	**1**
17	(a) 0.3	**1**
18	(d) Any point on the line segment joining the points (0,2) and (3,0)	**1**
	(ASSERTION-REASON BASED QUESTIONS)	
19.	**(a) Both A and R are true and R is the correct explanation of A.**	**1**
20.	**(d) A is false but R is true.**	**1**
	(Section B)	
21.	$f(n)=\begin{cases}\frac{n+1}{2}, & if\ n\ is\ odd \\ \frac{n}{2}, & if\ n\ is\ even\end{cases}$ $for\ all\ n\in N$ $f:N\rightarrow N$ is defined as it can be observed that : f(1)= $\frac{1+1}{2}=1$ and f(2)= $\frac{2}{2}=1$ Therefore f is not one -one . Consider a natural number (n) in co domain N Case -I :n is odd Thererfore n=2r+1 for some $r\in N$ then there exist $4r+1\in N$ such that f(4r+1)= $\frac{4r+1+1}{2}=2r+1$ Case II : nis even n=2r for some $r\in N$ then there exist $4r\in N$ such that f(4r)= $\frac{4r}{2}=2r$ therefore f is onto Hence f is not a bijective function **OR** $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)\neq\frac{13\pi}{6}$ as the range of principal value branch of $\cos^{-1}$ is $[0,\pi]$ So $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$ $=\cos^{-1}\left[\cos\frac{\pi}{6}\right]=\frac{\pi}{6}$ $\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\frac{\pi}{6}$	**1** **1** **1** **1**

22.	Let AB be the lamp post and let MN be the man of height 2m and let AM= l metre and MS be the shadow of the man ' Let length of shadow MS = s Given : Man walks at a speed of 5 k mph. Therefore dl/dt =5kmph We need to find the rate at which lenghth of his shadow increases =ds/dt In triagle ASB , $\tan\theta=\frac{AB}{AS}=\frac{6}{l+s}$ -----(1) in triangle MSN . $\tan\theta=\frac{MN}{MS}=\frac{2}{s}$ ------(2) From (1) and (2) $\frac{6}{l+s}=\frac{2}{s}$ $6s=2l+2s$ => $\frac{dl}{dt}=2\frac{ds}{dt}$ => $5=2\frac{ds}{dt}$ since it is given that $\frac{dl}{dt}=5\,kmph$ => $\frac{ds}{dt}=\frac{5}{2}kmph$ =2.5 k mph 	1 1
23.	Given $\|\vec{a}\|=4\hat{i}+3\hat{j}+\hat{k}$ and $\|\vec{b}\|=2\hat{i}-\hat{j}+2\hat{k}$ Hence $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 3 & 1\\ 2 & -1 & 2\end{vmatrix}=\hat{i}(6+1)-\hat{j}(8-2)+\hat{k}(-4-6)=7\hat{i}-6\hat{j}-10\hat{k}$ Unit vector perpendicular to each of the vector $\vec{a}$ and $\vec{b}$ $=\frac{\vec{a}\times\vec{b}}{\lceil\vec{a}\times\vec{b}\rceil}=\frac{7\hat{i}-6\hat{j}-10\hat{k}}{\sqrt{(7)^2+(-6)^2+(-10)^2}}=\frac{7}{\sqrt{185}}\hat{i}-\frac{6}{\sqrt{185}}\hat{j}-\frac{10}{\sqrt{185}}\hat{k}$ **OR** D.r'.s of the line through A(4 ,1 ,2) and B (5 λ , 0) are 5 - 4 , - $\lambda-1$, 0 -2 i.e. 1 $\lambda-1$, -2 D.r'.s of the line through C(2 ,1 ,1) and D(3 , 3 ,-1) are 3 - 2 , - $3-1$, -1 -1 i.e. 1 , 2 -2 Because line AB|| line CD therefore D,r.'s are proportional . therefore $\frac{1}{1}=\frac{\lambda-1}{2}=\frac{-2}{-2}$ or $1=\frac{\lambda-1}{2}=1$ => $\lambda-1=2$ => $\lambda=3$ **OR** Direction ratios of line joining A and B are 1-2 ,-2-3 , 3+4 i .e. -1 ;5 .7 Direction ratios of line joining B and C are 3-1 ,8+2 , -11 -3 i .e. 1 ,5 .7 i.e. 2 ,10 ,-14	1 1 1 1

	It is clear that direction ratios of AB and BC are proportional , Hence AB is parallel to BC . But point B is common to both AB and BC . Therefore A ,B ,C ARE collinear points. Thus $\overrightarrow{OA} = \vec{a} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\overrightarrow{OB} = \vec{b} = x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, $\overrightarrow{OP} = \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$	
24.	We have $If\ x\sqrt{1+y}+y\sqrt{1+x}=0 \Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$ $\Rightarrow x^2(1+y)=-y^2(1+x)$ $\Rightarrow (x^2-y^2)+xy(x-y)=0$ $\Rightarrow y=\frac{-x}{x+1}$ Therefore $\frac{dy}{dx}=\frac{(x+1)(-1)-(-x)(1)}{(x+1)^2} = \frac{-x-1+x}{(x+1)^2} = \frac{-1}{(x+1)^2}$	**1** **1**
25.	Since $\vec{a}$ is unit vector $\therefore \Rightarrow \|\vec{a}\|=1$ $(\vec{x}-\vec{a})(\vec{x}+\vec{a})=15 \Rightarrow \vec{x}.\vec{x}-\vec{x}.\vec{a}+\vec{a}.\vec{x}-\vec{a}.\vec{a}=15 \Rightarrow \|\vec{x}\|^2-\|\vec{a}\|^2=15$ $\Rightarrow \|\vec{x}\|^2-1=15$ $\Rightarrow \|\vec{x}\|^2=16$ $\Rightarrow \|\vec{x}\|=\pm 4=4$	**1/2** **1/2** **1**
	Section c	
26 .	Let I= $\int \frac{dx}{\sqrt{(x-a)(x-b)}} = \int \frac{dx}{\sqrt{x^2-(a+b)x+ab}}$ $= \int \frac{dx}{\sqrt{x^2-2(\frac{a+b}{2})x+(\frac{a+b}{2})^2+ab-(\frac{a+b}{2})^2}} = \int \frac{dx}{\sqrt{\left(x-(\frac{a+b}{2})\right)^2-(\frac{a-b}{2})^2}}$ $= \log\left\|\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-(\frac{a+b}{2})\right)^2-(\frac{a-b}{2})^2}\right\| +C$ $= \log\left\|\left(x-\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right\| +C$	**2** **1**
27	Let event A_1 = transfered ball is red A_2 =transfered ball is black b=ball drawn from bag II is red $P(A_1)=\frac{3}{7} \quad P(A_2)=\frac{4}{7} \quad P(\frac{B}{A_1})=\frac{5}{10} \quad P(\frac{B}{A_2})=\frac{4}{10}$ $P(\frac{A_2}{B})=\frac{P(A_2)P(\frac{B}{A_2})}{P(A_1)P(\frac{B}{A_1})+P(A_2)P(\frac{B}{A_2})} = \frac{\frac{4}{7}x\frac{4}{10}}{\frac{4}{7}x\frac{4}{10}+\frac{3}{7}x\frac{5}{10}} = \frac{16}{31}$ **OR** Let the number of red card in a sample of 3 cards X=0, 1, 2, 3	$1\frac{1}{2}$ $1\frac{1}{2}$

	P(X=0)= $\frac{^{26}C_3}{^{52}C_3} = \frac{2}{17}$ P(X=1)= $\frac{^{26}C_1\,^{26}C_2}{^{52}C_3} = \frac{13}{34}$ P(X=2)= $\frac{^{26}C_2\,^{26}C_1}{^{52}C_3} = \frac{13}{34}$ P(X=3)= $\frac{^{26}C_3}{^{52}C_3} = \frac{2}{17}$ The Probability distribution is as follows	
	X: 0, 1, 2, 3 P(X): $\frac{2}{17}$, $\frac{13}{34}$, $\frac{13}{34}$, $\frac{2}{17}$	2
	$\sum(x) = 0\,x\,\frac{2}{17} + 1\,x\,\frac{13}{34} + 2\,x\,\frac{13}{34} + 3\,x\,\frac{2}{17}$ $\sum(x) = \frac{3}{2}$	1
28.	Let I= $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{sinx}+\sqrt{\cos x}}\,dx$ ------(i) = $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$	1
	= $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ --------(ii) Adding (I) and (ii) we get 2I= $\int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\sin x}}{\sqrt{sinx}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{cosx}+\sqrt{\sin x}}\right] dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$	1
	= $\int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ => I= $\frac{\pi}{4}$	1
	OR Let I= $\int_2^8 \lvert x-5\rvert\,dx = -\int_2^5 \lvert x-5\rvert\,dx + \int_5^8 \lvert x-5\rvert\,dx$	

X	0	1	2	3
P(X)	$\frac{2}{17}$	$\frac{13}{34}$	$\frac{13}{34}$	$\frac{2}{17}$

	$\therefore \quad \lvert x-5\rvert = \begin{cases} -(x-5) & if\ x<5 \\ x-5 & if\ x\geq 5 \end{cases}$ $\therefore$ I= $-\int_2^5 (x-5)dx+\int_5^8 (x-5)dx$ = $-\left\{\frac{x^2}{2}-5x\right\}\Big\vert_2^5 + \left\{\frac{x^2}{2}-5x\right\}\Big\vert_5^8$ = $\left\{\frac{(-2+2)^2}{2}-\frac{(-5+2)^2}{2}\right\} + \left\{\frac{(5+2)^2}{2}-\frac{(-2+2)^2}{2}\right\}$ = $\frac{1}{2}(25-4)+5(5-2)+\frac{1}{2}(64-25)-5(8-5)$ = $\frac{-21}{2}+15+\frac{+39}{2}-15$ = $\frac{18}{2}$ = 9	1 2
sol29/2	The given D.E is $ydx-(x+2y^2)dy=0$ dividing each term by dy we get $y\frac{dx}{dy}-x-2y^2=0$ or $y\frac{dx}{dy}-x=2y^2$ dividing by y $\frac{dx}{dy}-\frac{x}{y}=2y$ It is of the form $\frac{dx}{dy}+Px=Q$ compairing P=-1/y ,Q=2y $\int P\,dy=-\int\frac{1}{y}dy=-\log y$ I. F;= $e^{\int Pdy}$ = e^{-logy} = $e^{\log y^{-1}}$ = y^{-1} = $\frac{1}{y}$ Therefore general solution is $x(I.F)=\int Q.(I.F)dy+C$ or $x.\frac{1}{y}=\int 2y.\frac{1}{y}dy+C$ = $\int 2.dy+C$ =2y+c Multiplying by y => x=y(2y+c) **OR** Given differential equation $(x-y)dy-(x+y)dx=0$ can be written as $\frac{dy}{dx}=\frac{(x+y)}{(x-y)}=\frac{1+(\frac{y}{x})}{1-(\frac{y}{x})}$ ------(1) since RHS is of the form g(y/x) so it is homogeneous function of degree zero Therefore equation (1) is a homogeneous differential equation. To solve this put y=vx ------(ii) Differentiating (ii) w r t x we get $v+x\frac{dv}{dx}=\frac{(1+v)}{(1-v)}$ => $x\frac{dv}{dx}=\frac{(1+v)}{(1-v)}-v$	 1 2 1

=> $x\frac{dv}{dx}=\frac{(1+v-v+v^2)}{(1-v)}$

=> $x\frac{dv}{dx}=\frac{(1+v^2)}{(1-v)}$

integrating both sides we get

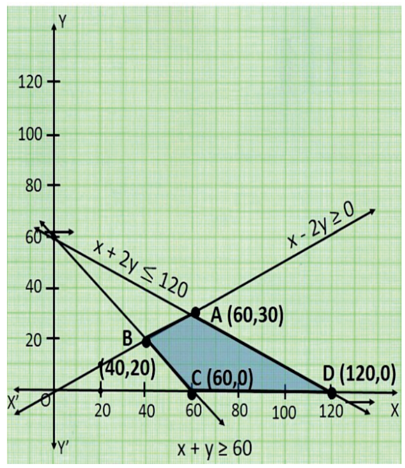
$\int\frac{dx}{x} = \int\frac{(1-v)}{(1+v^2)}dv$ => $\int\frac{dx}{x} = \int\frac{1}{(1+v^2)}dv - \int\frac{2v}{(1+v^2)}dv$ **1**

=>log x +c = $\tan^{-1}v - \frac{1}{2}\log(1+v^2)$ =>l $logx+\frac{1}{2}\log(1+\frac{y^2}{x^2})+C=\tan^{-1}\frac{y}{x}$

from (ii) $\frac{1}{2}\log(x^2+y^2)+C=\tan^{-1}\frac{y}{x}$ **1**

30/2 Sol

Minimize Z=5x+10y
Subject to constraints



$x+2y\leq 120$

x	0	120
y	60	0

$x+y\geq 60$

x	60	0
y	0	60

$x-2y\geq 0$

x	0	20
y	0	10

1

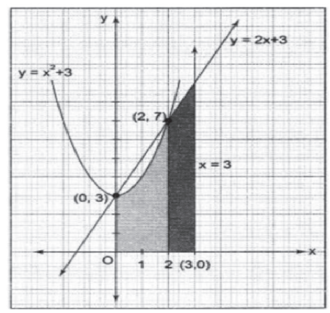
Corner points	Value of Z
(60 ,30)	600 ← Maximum Value
B(40 ,20)	450
C(60 ,0)	**300 ← Minimum Value**
D (120 ,0)	600 ← Maximum Value

2

Therefore Z=300 is minimum at (60, 0)

31. Let I= $\int\frac{1}{x(x^4-1)}dx = \frac{1}{4}\int\frac{4x^3dx}{x(x^4-1)}$ **1**

<table>
<tr><td></td><td>put x^4 =t => $4x^3dx=dt$
$\therefore$ I=$\frac{1}{4}\int\frac{dt}{t(t-1)}$
We write $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{(t-1)}$ => 1=A(t-1)+Bt
putting t=0 in (I) we get 1=A(-1) => A=-1
Putting t=1 in (I) we get 1=B(1) => B=-1
$\therefore$ $\frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{(t-1)}$
=>I= $\frac{1}{4}\int(\frac{-1}{t}+\frac{1}{t-1})dt$ = $\frac{1}{4}\int(\frac{-1}{t}+\frac{1}{t-1})dt$= $\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C$ = $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$</td><td>1

1</td></tr>
<tr><td></td><td><u>(SECTION D)</u></td><td></td></tr>
<tr><td>32.</td><td>

The points of intersection of the parabola $y=x^2+3$ and line y=2x+3 are (0, 3) and (2, 7)
Required area = $\int_0^2 x^2+3$ + $\int_2^3(2x+3)$
= $[\frac{x^3}{3}+3x]\Big|_0^2$ + $[\frac{2x^2}{2}+3x]\Big|_2^3$ = $\frac{8}{3}+6+9+9-(4+6)$
= $\frac{8}{3}+24-10$ = $\frac{50}{3}$ sq .units.</td><td>Correct fig. 1 mark
$\frac{1}{2}$
2
$1\frac{1}{2}$</td></tr>
<tr><td>33.</td><td>We have a relation R in the set A ={ 1, 2, 3, 4, 5} defined as R={ (a, b):|a-b| is divisible by 2}
Clearly , R ={ (1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5),(4, 2), (4, 4), (5, 1), (5, 3), (5, 5)}
Reflexive: For any $a\in A$, we have |a -a| =0, which is divisible by 2.
⇒ $(a,a)\in R\forall a\in A$.
Thus R is reflexive
Symmetric : Let a, b, A such that $(a,b)\in R$
⇒ |a-b| is divisible by 2
⇒ $|a-b|=2\lambda$ for some $\lambda\in N$
⇒ $|b-a|=2\lambda$ for some $\lambda\in N$ {because |a-b|=|b-a|}
⇒
$(b,a)\in R$
Thus R is symmetric

Transitive: let $a,b,c\in R$ such that $(a,b,)\in R$ and $(b,c)\in R$

⇒ |a-b| is divisible by 2 and |b-c| is divisible by 2
⇒ $|a-b|=2\lambda$ and $|b-c|=2\mu$ for some $\lambda,\mu\in N$
Now $|a-c|=|(a-b)+(b-c)|$
⇒ $a-b=\pm2\lambda$ and $b-c=\pm2\mu$
$|a-c|=|(a-b)+(b-c)|$ = $|\pm2\lambda+(\pm2\mu)|$ = $|\pm2\lambda\pm2\mu|$ =2 $|\pm\lambda\pm\mu|$ =2 [some positive number]
⇒ $|a-c|$ is divisible by 2 ⇒ $(a,c)\in R$
Thus , R is transitive
Hence R is an equivalence relation .</td><td>1

1

1

1

1</td></tr>
</table>

OR

Given that A={ 2, 3, 4} , B= { 2, 5, 6, 7 **2+2+2**

(a)Let $f : A \to B$ denotes a mapping

f={ (x, y):y=x+3}

or f={ (2, 5), (3, 6), (4, 7)} which is an injective mapping

(b) Let $g : A \to B$ denote a mapping such that g={ (2, 2)(3, 5) (4, 5) which is not an injective mapping [Because 5 has two pre images]

(c) Let $h : B \to A$ denote a mapping such that h={ (2, 2)(5, 3) (6 4)(7, 4)which is one of the mapping from B to A

---

34. From the given equation we get $\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t(-\hat{i} + \hat{j} - 2\hat{k})$ --------(1)

and $\vec{r} = \hat{i} - \hat{j} - \hat{k} + s(\hat{i} + 2\hat{j} - 2\hat{k})$ -------(ii) **1**

Equation (I) and (ii) are of the form

$\vec{r} = \vec{a}_1 + t\vec{b}_1$ and $\vec{r} = \vec{a}_2 + s\vec{b}_2$,

where $\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$ **1**

$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

here $\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$

Also $\vec{b}_1$ x $\vec{b}_1$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix}$ = $(-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k}$ = $2\hat{i} - 4\hat{j} - 3\hat{k}$ **1**

$|\vec{b}_1 x \vec{b}_1| = \sqrt{2^2 + (-4)^2 + (-3)^2}$ = $\sqrt{2^2 + (-4)^2 + (-3)^2}$ = $\sqrt{4+16+9}$ = $\sqrt{4+16+9}$ = $\sqrt{29}$ **1**

Required shortest distance = $\left| \frac{(\vec{b}_1 x \vec{b}_1).(\vec{d}_2 - \vec{d}_1)}{|\vec{b}_2 x \vec{b}_1|} \right|$ = $\left| \frac{(2\hat{i} - 4\hat{j} - 3\hat{k}).(\hat{j} - 4\hat{k})}{\sqrt{29}} \right|$ = $\frac{8}{\sqrt{29}}$ units **1**

OR

It is given that equation of the line as $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ and z=

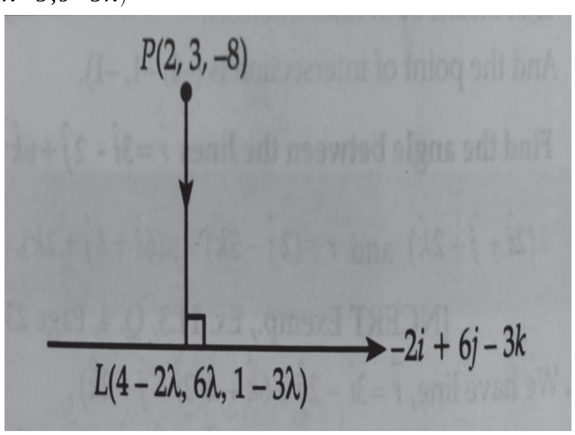
=> $\frac{4-x}{-2} = \frac{y}{6} = \frac{1-z}{-3} = \lambda$ => $x = -2\lambda + 4$ $y = 6\lambda$ $z = -3\lambda + 1$ **1**

Let the foot of perpendicular om the point P(2, 3, -8) on the line is L (4-2 $L(4-2\lambda, 6\lambda, 1-3\lambda)$ **1**

Then the direction ratios of PL are proportional to

$(4-2\lambda-2, 6\lambda-3, 1-3\lambda+8)$

or $(2-2\lambda, 6\lambda-3, 9-3\lambda)$ **1**



1

1

	Also direction ratios of line are -2, 6, -3. Since , Pl is perpendicular to the given line $\therefore \quad -2(2-2\lambda)+6(6\lambda-3)-3(9-3\lambda)=0$ => $-4+4\lambda+36\lambda-18-27+9\lambda=0$ => $49\lambda=49$ => $\lambda=1$ So the coordinates of L are $L(4-2\lambda, 6\lambda, 1-3\lambda)\equiv(2,6,-2)$ Also length of PL = $\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}$ = $\sqrt{0+9+36}$ = $3\sqrt{5}$ units.	
35.	We have A=$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ --------------(i) $\therefore \quad \|A\|=3(-2)+5(1)=-1\neq 0$ Now $A_{11}=0, A_{12}=-(-2)=2, A_{13}=1$ $A_{21}=-1, A_{22}=-9, A_{23}=-5$ $A_{31}=2, A_{32}=23, A_{33}=13$ $\therefore$ adjA = $\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$ $\therefore \quad A^{-1}=\frac{adj.A}{\|A\|}=\frac{1}{1}=\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$ => $A^{-1}=\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$ Also we have system of equations as 2x – 3y+5z =11 3x +2y -4z=5 and x+y-2z=7 in the form of AX=B,=> $X=A^{-1}B$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$= $\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$ $\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$ On solving we get x=1 ,y=2, z=3	**1** **1** **1** **1** **1**

	(Section E)	
36.	Sol(i)Area = length x breadth =>A=xy Since P=2(x+y) => $\frac{P-2x}{2}=y$ (ii) We have $A=\frac{Px-2x^2}{2}$ $\frac{dA}{dX}=\frac{1}{2}(P-4x)=0$ => $P-4x=0$ => $x=\frac{P}{4}$ Clearly at $x=\frac{P}{4}$ $\frac{d^{2A}}{dX^2}=-2<0$ therefore area is maximum at $x=\frac{P}{4}$ (iii)) We have $y=\frac{P-2x}{2}=\frac{P}{2}-\frac{P}{4}=\frac{P}{4}$ or $A=xy=\frac{P}{4}x\frac{P}{4}=\frac{P^2}{4}$	**1+1+2**
37	(i) $x^{2y}=1024$ (ii) $C=10x^2+10xy$ (iii) $C=10x^2+\frac{10240}{x}$ **OR** (v)1920	**1** **1** **2**
38.	E_1 : Seed of the type A_1 E_2 : Seed of the type A_2 E_3 : Seed of the type A_3 A Seed germinates. (i)P(A)= $P(E_1)P(\frac{A}{E_1})+P(E_2)P(\frac{A}{E_2})+P(E_3)P(\frac{A}{E_3})$ = $\frac{4}{10}x\frac{45}{100}+\frac{4}{10}x\frac{60}{100}+\frac{2}{10}x\frac{35}{100}$ = $\frac{180+240+70}{1000}$ = $\frac{490}{1000}$ = $\frac{49}{100}$ (b) $P(\frac{E_2}{A})=\frac{P(E_2)P(\frac{A}{E_2})}{P(A)}$ = $\frac{\frac{4}{10}x\frac{60}{100}}{\frac{490}{1000}}$ = $\frac{240}{490}$ = 24/49	**1 $\frac{1}{2}$** **1/2** **1** **1**

DIRECTORATE OF EDUCATION, GNCT OF DELHI

PRACTICE PAPER – 2

(SESSION 2022 – 23)

CLASS XII

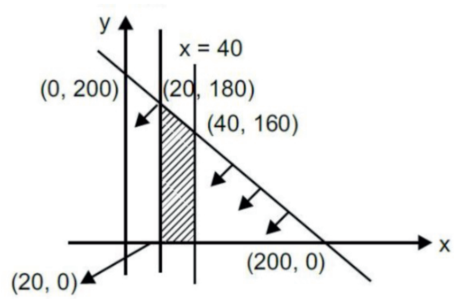
MATHEMATICS (CODE: 041)

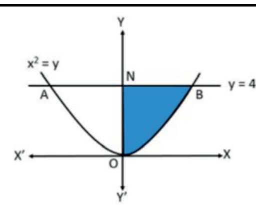
Time Allowed: 3 HOURS **Maximum Marks: 80**

General Instructions:

1. This question paper contains **FIVE sections – A, B, C, D & E**. Each part is compulsory. However, there are internal choices in some questions.
2. **Section A has 18 MCQ's and 02 Assertion-Reason based questions of 1 mark each.**
3. **Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.**
4. **Section C has 6 Short Answer (SA)-type questions of 3 marks each.**
5. **Section D has 4 Long Answer (LA)-type questions of 5 marks each.**
6. **Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.**

	SECTION – A **(Multiple Choice Questions)** **Each question carries 1 mark** **Each MCQ has four options with only one correct option, choose the correct option.**	
1.	*A set of values of decision variables that satisfies the linear constraints and non – negativity conditions of an L.P.P. is called its:* (a) *Unbounded solution* (b) *Feasible solution* (c) *Optimum solution* (d) *None of these*	1
2.	*The value of the expression* $\operatorname{cosec}^{-1}(2) + \cos^{-1}(\frac{1}{2}) + \tan^{-1}(-1)$ *is* $(a)\ \frac{\pi}{4}$ $(b)\ \frac{\pi}{3}$ $(c)\ \frac{\pi}{6}$ $(d)\ \frac{\pi}{2}$	1
3.	$If\ A=\begin{pmatrix}1&2\\2&1\end{pmatrix}, B=\begin{pmatrix}1\\4\\0\end{pmatrix} \& C=\begin{pmatrix}0&0&1\\3&1&2\end{pmatrix}$, *then* order of Matrix P is, where $P = ACB$ $(a)\ 2\times1$ $(b)\ 2\times3$ $(c)\ 2\times2$ $(d)\ 3\times2$	1

4.	*If A is square matrix of order* 3×3 *such that* $\lvert adj\ A\rvert = 16$, *such that* $(\lvert 2A\rvert)^2 = 2^p$, *then the* $p =$ (a) 4 (b) 5 (c) 8 (d) 10	1
5.	For an L.P.P. the objective function is Z = 400x + 300y, and the feasible region determined by a set of constraints (linear inequations) is shown in the graph.Coordinates at which the objective function is maximum is  (a) (20, 0) (b) (40, 0) (c) (40, 160) (d) (20, 180)	1
6.	If A and B are two independent events with $P(A) = \frac{3}{5}$ & $P(B) = \frac{4}{9}$, then $9P(\overline{A}\cap\overline{B}) =$ (a) 1 (b) 2 (c) 3 (d) 4	1
7.	*If* $\vec{a}$ *is a unit vector and* $(\vec{x}+\vec{a}).(\vec{x}-\vec{a}) = 15$, *then* $\lvert\vec{x}\rvert =$ (a) 1 (b) 2 (c) 3 (d) 4	1
8.	*Suppose P, Q and R are different matrices of order* 3×5, $a\times b$ *and* $c\times d$ *respectively, then value of* $ac + bd$ *is, if matrix* $2P + 3Q - 4R$ *is defined* (a) 9 (b) 30 (c) 34 (d) 15	1
9.	*If A is a symmetric matrix then which of the following is not Symmetric matrix,* $(a)\ A + A^T$ $(b)\ A.A^T$ $(c)\ A - A^T$ $(d)\ A^T$	1
10.	*If* $y^{\frac{1}{x}} = a$, *then* $\frac{dy}{dx} =$ $(a)\ y$ $(b)\ ay$ $(c)\ ax$ $(d)\ y(\log_e a)$	1

11.	*If a non-singular Matrix A satisfy* $2A^2 + A - I = O$, *then* $A^{-1} =$ $(a)\ 2A - I$ $(b)\ 2A + I$ $(c)\ 4A + 2I$ $(d)\ 2A - 4I$	1
12.	$If\ \lvert\vec{a}\rvert = 13, \lvert\vec{b}\rvert = 1\ and\ \lvert\vec{a}.\vec{b}\rvert = 12, then\ \lvert\vec{a}\times\vec{b}\rvert =$ (a) 5 (b) 4 (c) 1 (d) 3	1
13.	*The* sum of order & degree of the differential equation $\frac{d}{dx}\left(\left(\frac{d^2y}{dx^2}\right)^3\right) = 0$ is (a) 5 (b) 4 (c) 2 (d) 3	1
14.	*If the sum of the two unit vectors is a unit vector, then the magnitude of their difference is* (a) 3 (b) $\sqrt{3}$ (c) 1 (d) $\sqrt{2}$	1
15.	*If* $(2\hat{i} + 6\hat{j} + 9\hat{k}) \times (\hat{i} + p\hat{j} + q\hat{k}) = \vec{0}$, *then* $p + 2q =$ $(a)10$ $(b)11$ $(c)12$ $(d)13$	1
16.	*If a line makes an angle* α, β, γ with the axes respectively, *then* $\cos 2\alpha + \cos 2\beta + \cos 2\gamma =$ $(a)\ -1$ $(b)\ 0$ $(c)\ 1$ $(d)\ 2$	1
17.	*The* general solution of the differential equation $\frac{dx}{dy} = \frac{x}{y}$ is $(a)\ x = y + c$ $(b) x - y = c$ $(c) xy = c$ $(d) \frac{x}{y} = c$	1
18.	*What* is the shaded *Area* (in sq. units) shown in the figure given : $(a)\ \frac{16}{3}$ $(b)\ \frac{32}{3}$ $(c)\ \frac{8}{3}$ $(d)\ \frac{4}{3}$ 	1

	ASSERTION-REASON BASED QUESTIONS (Q.19 & Q.20) **In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.** **(a) Both A and R are true and R is the correct explanation of A.** **(b) Both A and R are true but R is not the correct explanation of A.** **(c) A is true but R is false.** **(d) A is false but R is true.**	
19.	$ASSERTION(A): \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^{2021} + \sin^{2023} x + 1)dx = 0$ $REASONING(R): \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if\ f(-x) = f(x) \\ 0 \quad ,if\ f(-x) = -f(x) \end{cases}$	1
20.	$ASSERTION(A): If\ sin(x+y) + \cos(x+y) = 1,\ then \frac{dy}{dx} = -1$ $REASONING(R): The\ derivative\ of\ an\ odd\ function\ is\ always\ an\ even\ function$	1

SECTION B

This section comprises of very short answer type-questions (VSA) of 2 marks each

21.	*Find* the principal value of $\sin^{-1}(\sin \frac{3\pi}{5})$. *OR* *A relation R in the set of real numbers R is given by* $R = \{(a,\ b): a > b,\ such\ that\ a, b \in R\}$. *Check* the Transitivity of Relation R.	2
22.	*For the curve* $y = 5x - 2x^3$, *if x increases at the rate of* 2 *units / sec, then how fast is the slope of curve changing when* $x = 3$?	2

23.	*If* $\sin y = x\sin(a+y)$, *then* prove that $\boxed{\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}}$. *OR* *If* $5^x + 5^y = 5^{x+y}$, *then* prove that $\boxed{\dfrac{dy}{dx} = -5^{y-x}}$.	2
24.	*If* $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ *and* $\vec{b} = \hat{i} - \hat{j} - \hat{k}$, *then Find a unit vector OPPOSITE* to *the direction of* $(\vec{a} + \vec{b})$.	2
25.	*Find the Direction* cos*ines of the line* $\dfrac{x-1}{1} = \dfrac{2y-6}{4} = \dfrac{4-z}{1}$.	2
SECTION C **(This section comprises of short answer type questions (SA) of 3 marks each)**		
26.	*Solve* the following Differential Equation: $\left(x^2 - y^2\right)dx + 2xy\ dy = 0$ *OR* *Find the general solution of the differential equation* $ydx - \left(x + 2y^2\right)dy = 0$	3
27.	*Find* the intervals in which $f(x) = \sin x + \cos x, x \in [0, 2\pi]$ is (a) strictly Increasing (b) strictly Decreasing	3
28.	*If* $x^p . y^q = (x+y)^{p+q}$, *Prove that* $\boxed{\dfrac{dy}{dx} = \dfrac{y}{x}}$. *OR* *If* $x\sqrt{1+y} + y\sqrt{1+x} = 0, x \neq y$, *then* prove that $\boxed{\dfrac{dy}{dx} = \dfrac{-1}{(1+x)^2}}$.	3

29.	$Evaluate: I = \int \frac{e^x.dx}{e^{2x} - 4e^x + 5}$ *OR* $Evaluate: I = \int_{-3}^{5} \lvert x - 2 \rvert dx$	3
30.	Solve the following Linear Programming Problem graphically: Minimise Z = 13x $-15y$ subject to the constraints $x + y = 7,\ 2x - 3y + 6 \geq 0, x \geq 0, y \geq 0$	3
31.	In a group of 50 scouts in a camp, 30 are well trained in first aid techniques while the remaining are well trained in hospitality but not in first aid. Two scouts are selected at random from the group. Find the probability distribution of number of selected scouts who are well trained in first aid. OR In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20%read both Hindi and English newspaper. A student is selected at random. (a) Find the probability that the student reads neither Hindi nor English newspaper. (b) If she reads Hindi newspaper, find the probability that she reads English newspaper. (c) If she reads English newspaper, find the probability that she reads Hindi newspaper.	3
	SECTION D **(This section comprises of long answer-type questions (LA) of 5 marks each)**	
32.	Show that each of the relation R in the set $A = \{x \in Z : 0 \leq x \leq 12\}$, given by $R = \{(a,b) : \lvert a - b \rvert$ is a multiple of 5} Find the set of all elements related to 1 in each case.	5

	OR *Let* $A = R - \{3\}$ *and* $B = R - \{1\}$. *Consider the function* $f : A \to B$ *defined by* $f(x) = \frac{x-2}{x-3}$. *Is* f *one* − *one and onto*? *Justify your answer.*	
33.	*Find the inverse of the matrix* $\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$ *and hence solve* *the system of equations*: $3x + 4y + 5z = 18$ $5x - 2y + 7z = 20$ $2x - y + 8z = 13$	5
34.	*Find the coordinates of the foot of perpendicular drawn from point* $P(1, 0, 3)$ *to the line joining the points* $A(4, 7, 1)$ *and* $B(3, 5, 3)$. OR *What* do you mean by Skew-lines. Find the shortest distance between following skew-lines: $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ *and* $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$	5
35.	*Find the area enclosed by the parabola* $4y = 3x^2$ *and the line* $2y = 3x + 12$. OR *Find the area of the region bounded by the line* $3x - 2y + 6 = 0$, *the* $x -$ *axis*, $x = -3$ *and* $x = 2$.	1

SECTION - C

This section comprises of 3 case-study/passage-based questions of 4 marks each with two sub-parts.
First two case study questions have three sub parts (A), (B) & (C) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. A poster is to be formed by the Government to promote the event on **G-20 Summit in India.** The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster (that is, the area between the margins) is fixed at 384 cm^2. **[4]**

(A) If a cm be the width and b cm be the height of the poster, then Expressed the area of the poster in terms of a and b.

(B) If a cm be the width and b cm be the height of the poster, then Expressed the area of the poster in terms of a only.

(C) Find the values of a & b, so that area of the poster is minimized.

37. An owner of a car rental company have determined that if they charge customers Rs x per day to rent a car, where $50 \leq x \leq 200$, then number of cars (n), they rent per day can be shown by linear function $n(x) = 1000 - 5x$. If they charge Rs. 50 per day or less they will rent all their cars. If they charge Rs. 200 or more per day they will not rent any car. **[4]**

	Based on the above information, answer the following question. (i) If R(x) denote the revenue, then find the value of x at which R(x) has maximum value. (ii) Find the Maximum revenue collected by company. OR Find the number of cars rented per day, when x = 75.	
38.	Between students of class XII of two schools A and B basketball match is organised. For which, a team from each school is chosen, say T_1, be the team of school A and T_2, be the team of school B. These teams have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probability of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{3}{10}$ & $\frac{1}{5}$ respectively. Each team gets 2 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by team A and B respectively, after two games.  (a) Find the value of $P(X > Y)$. (b) Find the value of $P(X + Y = 8)$.	4 2 2

NOTES

NOTES

NOTES