Reading Material - Mathematics

Class XI (2025-26)

Subject Code - 041

Chapter 1: Sets

Practical problems on Union and Intersection of two sets

1.12 Practical Problems on Union and Intersection of Two Sets

In earlier Section, we have learnt union, intersection and difference of two sets. In this Section, we will go through some practical problems related to our daily life. The formulae derived in this Section will also be used in subsequent Chapter on Probability (Chapter 16).

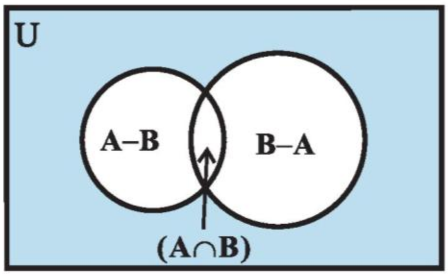

Fig 1.11

Let A and B be finite sets. If $A \cap B = \phi$, then

(i) $n(A \cup B) = n(A) + n(B)$ ... (1)

The elements in $A \cup B$ are either in A or in B but not in both as $A \cap B = \phi$. So, (1) follows immediately.

In general, if A and B are finite sets, then

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ... (2)

Note that the sets $A - B$, $A \cap B$ and $B - A$ are disjoint and their union is $A \cup B$ (Fig 1.11). Therefore

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$$
$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$
$$= n(A) + n(B) - n(A \cap B), \text{ which verifies (2)}$$

(iii) If A, B and C are finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \quad \text{... (3)}$$

In fact, we have

$$n(A \cup B \cup C) = n(A) + n(B \cup C) - n[A \cap (B \cup C)] \quad [\text{by (2)}]$$
$$= n(A) + n(B) + n(C) - n(B \cap C) - n[A \cap (B \cup C)] \quad [\text{by (2)}]$$

Since $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we get

$$n[A \cap (B \cup C)] = n(A \cap B) + n(A \cap C) - n[(A \cap B) \cap (A \cap C)]$$
$$= n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)$$

Therefore

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

This proves (3).

Example 23 If X and Y are two sets such that $X \cup Y$ has 50 elements, X has 28 elements and Y has 32 elements, how many elements does $X \cap Y$ have ?

Solution Given that

$$n(X \cup Y) = 50,\ n(X) = 28,\ n(Y) = 32,$$

$n(X \cap Y) = ?$

By using the formula

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y),$$

we find that

$$n(X \cap Y) = n(X) + n(Y) - n(X \cup Y)$$
$$= 28 + 32 - 50 = 10$$

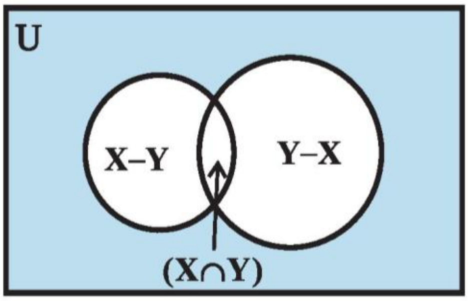

Fig 1.12

Alternatively, suppose $n(X \cap Y) = k$, then

$$n(X - Y) = 28 - k,\ n(Y - X) = 32 - k \text{ (by Venn diagram in Fig 1.12)}$$

This gives $50 = n(X \cup Y) = n(X - Y) + n(X \cap Y) + n(Y - X)$

$$= (28 - k) + k + (32 - k)$$

Hence $k = 10$.

Example 24 In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics ?

Solution Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach physics. In the statement of the problem, the word 'or' gives us a clue of union and the word 'and' gives us a clue of intersection. We, therefore, have

$$n(M \cup P) = 20,\ n(M) = 12 \text{ and } n(M \cap P) = 4$$

We wish to determine $n(P)$.

Using the result

$$n(M \cup P) = n(M) + n(P) - n(M \cap P),$$

we obtain

$$20 = 12 + n(P) - 4$$

Thus $n(P) = 12$

Hence 12 teachers teach physics.

Example 25 In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football ?

Solution Let X be the set of students who like to play cricket and Y be the set of students who like to play football. Then $X \cup Y$ is the set of students who like to play at least one game, and $X \cap Y$ is the set of students who like to play both games.

Given $n(X) = 24,\ n(Y) = 16,\ n(X \cup Y) = 35,\ n(X \cap Y) = ?$

Using the formula $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$, we get

$$35 = 24 + 16 - n(X \cap Y)$$

Thus, $n(X \cap Y) = 5$

i.e., 5 students like to play both games.

Example 26 In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice.

Solution Let U denote the set of surveyed students and A denote the set of students taking apple juice and B denote the set of students taking orange juice. Then

$$n(U) = 400,\ n(A) = 100,\ n(B) = 150 \text{ and } n(A \cap B) = 75.$$

Now
$$\begin{aligned} n(A' \cap B') &= n(A \cup B)' \\ &= n(U) - n(A \cup B) \\ &= n(U) - n(A) - n(B) + n(A \cap B) \\ &= 400 - 100 - 150 + 75 = 225 \end{aligned}$$

Hence 225 students were taking neither apple juice nor orange juice.

Example 27 There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to

(i) Chemical C_1 but not chemical C_2 (ii) Chemical C_2 but not chemical C_1

(iii) Chemical C_1 or chemical C_2

Solution Let U denote the universal set consisting of individuals suffering from the skin disorder, A denote the set of individuals exposed to the chemical C_1 and B denote the set of individuals exposed to the chemical C_2.

Here $n(U) = 200,\ n(A) = 120,\ n(B) = 50$ and $n(A \cap B) = 30$

(i) From the Venn diagram given in Fig 1.13, we have

$$A = (A - B) \cup (A \cap B).$$

$n(A) = n(A - B) + n(A \cap B)$ (Since A – B) and $A \cap B$ are disjoint.)

or $n(A - B) = n(A) - n(A \cap B) = 120 - 30 = 90$

Hence, the number of individuals exposed to chemical C_1 but not to chemical C_2 is 90.

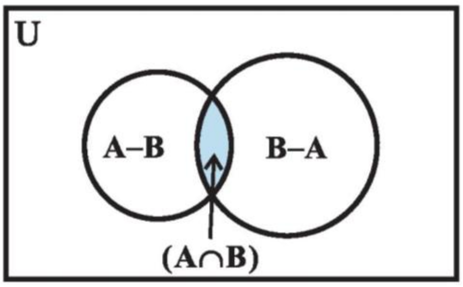

Fig 1.13

(ii) From the Fig 1.13, we have

$$B = (B - A) \cup (A \cap B).$$

and so, $n(B) = n(B - A) + n(A \cap B)$

(Since B – A and $A \cap B$ are disjoint.)

or $n(B - A) = n(B) - n(A \cap B)$
$= 50 - 30 = 20$

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Thus, the number of individuals exposed to chemical C_2 and not to chemical C_1 is 20.

(iii) The number of individuals exposed either to chemical C_1 or to chemical C_2, i.e.,

$$n (A \cup B) = n (A) + n (B) - n (A \cap B)$$

$$= 120 + 50 - 30 = 140.$$

EXERCISE 1.6

1. If X and Y are two sets such that $n (X) = 17$, $n (Y) = 23$ and $n (X \cup Y) = 38$, find $n (X \cap Y)$.
2. If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements and Y has 15 elements ; how many elements does $X \cap Y$ have?
3. In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many people can speak both Hindi and English?
4. If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have?
5. If X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, how many elements does Y have?
6. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?
7. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket? How many like tennis?
8. In a committee, 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak at least one of these two languages?

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Chapter 3: Trigonometric Functions

General solution of trigonometric equations of the type $\sin y = \sin a$, $\cos y = \cos a$ and $\tan y = \tan a$.

3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π. The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution*. We shall use '**Z**' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 18 Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.

Solution We know that, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Therefore, principal solutions are $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Example 19 Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.

Solution We know that, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

and $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

Thus $\tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$.

Therefore, principal solutions are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

We will now find the general solutions of trigonometric equations. We have already

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seen that:

$\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n + 1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$.

We shall now prove the following results:

Theorem 1 For any real numbers x and y,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \quad \text{or} \quad 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

which gives $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n + 1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n + 1)\pi + (-1)^{2n+1} y$ or $x = 2n\pi + (-1)^{2n} y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y, $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \quad \text{i.e.,} \quad -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

Thus $\sin\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd mulitple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or $$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin(x - y) = 0$ (Why?)

Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

Hence $\sin x = \sin\frac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

> **Note** $\frac{4\pi}{3}$ is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \frac{1}{2}$.

Solution We have, $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

Therefore $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or $$\tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$$

Therefore $$2x = n\pi + x + \frac{5\pi}{6}, \text{ where } n \in \mathbf{Z}$$

or $$x = n\pi + \frac{5\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

Example 23 Solve $\sin 2x - \sin 4x + \sin 6x = 0$.

Solution The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or $$2\sin 4x \cos 2x - \sin 4x = 0$$

i.e. $$\sin 4x(2\cos 2x - 1) = 0$$

Therefore $$\sin 4x = 0 \quad \text{or} \quad \cos 2x = \frac{1}{2}$$

i.e. $$\sin 4x = 0 \quad \text{or} \quad \cos 2x = \cos\frac{\pi}{3}$$

Hence $$4x = n\pi \quad \text{or} \quad 2x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbf{Z}$$

i.e. $$x = \frac{n\pi}{4} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

Example 24 Solve $2\cos^2 x + 3\sin x = 0$

Solution The equation can be written as

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

or $$2\sin^2 x - 3\sin x - 2 = 0$$

or $$(2\sin x + 1)(\sin x - 2) = 0$$

Hence $$\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = 2$$

But $\sin x = 2$ is not possible (Why?)

Therefore $$\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}.$$

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Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

EXERCISE 3.4

Find the principal and general solutions of the following equations:

1. $\tan x = \sqrt{3}$
2. $\sec x = 2$
3. $\cot x = -\sqrt{3}$
4. $\text{cosec } x = -2$

Find the general solution for each of the following equations:

5. $\cos 4x = \cos 2x$
6. $\cos 3x + \cos x - \cos 2x = 0$
7. $\sin 2x + \cos x = 0$
8. $\sec^2 2x = 1 - \tan 2x$
9. $\sin x + \sin 3x + \sin 5x = 0$

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Unit-II Algebra

Chapter: Principle of Mathematical Induction

Process of the proof by induction, motivating the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

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Chapter **4**

PRINCIPLE OF MATHEMATICAL INDUCTION

❖*Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted to it for his theorem of the binomial and the principle of universal gravity. – LAPLACE* ❖

4.1 Introduction

One key basis for mathematical thinking is deductive reasoning. An informal, and example of deductive reasoning, borrowed from the study of logic, is an argument expressed in three statements:

(a) Socrates is a man.
(b) All men are mortal, therefore,
(c) Socrates is mortal.

If statements (a) and (b) are true, then the truth of (c) is established. To make this simple mathematical example, we could write:

(i) Eight is divisible by two.
(ii) Any number divisible by two is an even number, therefore,
(iii) Eight is an even number.

G. Peano (1858-1932)

Thus, deduction in a nutshell is *given a statement to be proven, often called a conjecture or a theorem in mathematics, valid deductive steps are derived and a proof may or may not be established,* i.e., *deduction is the application of a general case to a particular case.*

In contrast to deduction, inductive reasoning depends on working with each case, and developing a conjecture by observing incidences till we have observed each and every case. It is frequently used in mathematics and is a key aspect of scientific reasoning, where collecting and analysing data is the norm. Thus, in simple language, we can say the word induction means the generalisation from particular cases or facts.

In algebra or in other discipline of mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements the well-suited principle that is used–based on the specific technique, is known as the *principle of mathematical induction*.

4.2 Motivation

In mathematics, we use a form of complete induction called mathematical induction. To understand the basic principles of mathematical induction, suppose a set of thin rectangular tiles are placed as shown in Fig 4.1.

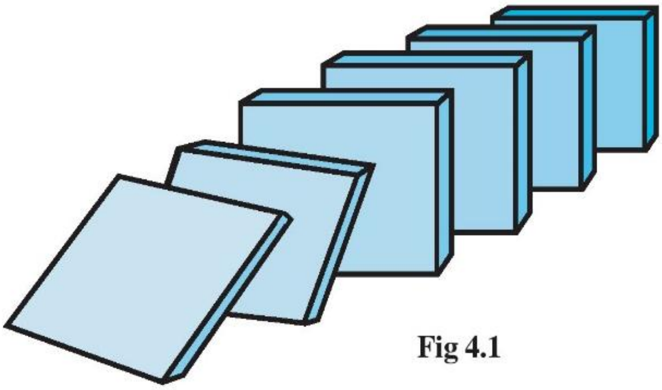

Fig 4.1

When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that

(a) The first tile falls, and

(b) In the event that any tile falls its successor necessarily falls.

This is the underlying principle of mathematical induction.

We know, the set of natural numbers **N** is a special ordered subset of the real numbers. In fact, **N** is the smallest subset of **R** with the following property:

A set S is said to be an inductive set if $1 \in$ S and $x + 1 \in$ S whenever $x \in$ S. Since **N** is the smallest subset of **R** which is an inductive set, it follows that any subset of **R** that is an inductive set must contain **N**.

Illustration

Suppose we wish to find the formula for the sum of positive integers 1, 2, 3,...,n, that is, a formula which will give the value of 1 + 2 + 3 when n = 3, the value 1 + 2 + 3 + 4, when n = 4 and so on and suppose that in some manner we are led to believe that the formula $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ is the correct one.

How can this formula actually be proved? We can, of course, verify the statement for as many positive integral values of n as we like, but this process will not prove the formula for all values of n. What is needed is some kind of chain reaction which will have the effect that once the formula is proved for a particular positive integer the formula will automatically follow for the next positive integer and the next indefinitely. Such a reaction may be considered as produced by the method of mathematical induction.

4.3 The Principle of Mathematical Induction

Suppose there is a given statement P(n) involving the natural number n such that

(*i*) *The statement is true for n* = 1, i.e., *P*(1) *is true, and*

(*ii*) *If the statement is true for n* = *k* (*where k is some positive integer*), *then the statement is also true for n* = *k* + 1, i.e., *truth of P*(*k*) *implies the truth of P* (*k* + 1).

Then, P(n) is true for all natural numbers n.

Property (i) is simply a statement of fact. There may be situations when a statement is true for all $n \geq 4$. In this case, step 1 will start from $n = 4$ and we shall verify the result for $n = 4$, i.e., P(4).

Property (ii) is a conditional property. It does not assert that the given statement is true for $n = k$, but only that if it is true for $n = k$, then it is also true for $n = k + 1$. So, to prove that the property holds , only prove that conditional proposition:

If the statement is true for $n = k$, then it is also true for $n = k + 1$.

This is sometimes referred to as the inductive step. The assumption that the given statement is true for $n = k$ in this inductive step is called the *inductive hypothesis*.

For example, frequently in mathematics, a formula will be discovered that appears to fit a pattern like

$$1 = 1^2 = 1$$
$$4 = 2^2 = 1 + 3$$
$$9 = 3^2 = 1 + 3 + 5$$
$$16 = 4^2 = 1 + 3 + 5 + 7, \text{ etc.}$$

It is worth to be noted that the sum of the first two odd natural numbers is the square of second natural number, sum of the first three odd natural numbers is the square of third natural number and so on.Thus, from this pattern it appears that

$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$, i.e,

the sum of the first n odd natural numbers is the square of n.

Let us write

P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.

We wish to prove that P(n) is true for all n.

The first step in a proof that uses mathematical induction is to prove that P (1) is true. This step is called the basic step. Obviously

$1 = 1^2$, i.e., P(1) is true.

The next step is called the *inductive step*. Here, we suppose that P (k) is true for some positive integer k and we need to prove that P $(k + 1)$ is true. Since P (k) is true, we have

$$1 + 3 + 5 + 7 + ... + (2k - 1) = k^2 \quad ... (1)$$

Consider

$$1 + 3 + 5 + 7 + ... + (2k - 1) + \{2(k + 1) - 1\} \quad ... (2)$$

$$= k^2 + (2k + 1) = (k + 1)^2 \quad \text{[Using (1)]}$$

Therefore, P $(k + 1)$ is true and the inductive proof is now completed.

Hence P(n) is true for all natural numbers n.

Example 1 For all $n \geq 1$, prove that

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution Let the given statement be P(n), i.e.,

$$P(n): \ 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$, $\quad P(1): 1 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1$ which is true.

Assume that P(k) is true for some positive integer k, i.e.,

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad ... (1)$$

We shall now prove that P$(k + 1)$ is also true. Now, we have

$$(1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2) + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[Using (1)]}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus P$(k + 1)$ is true, whenever P (k) is true.

Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Example 2 Prove that $2^n > n$ for all positive integers n.

Solution Let P(n): $2^n > n$

When $n = 1$, $2^1 > 1$. Hence P(1) is true.

Assume that P(k) is true for any positive integer k, i.e.,

$$2^k > k \qquad \text{... (1)}$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

Multiplying both sides of (1) by 2, we get

$$2.\ 2^k > 2k$$

i.e., $$2^{k+1} > 2k = k + k > k + 1$$

Therefore, P(k + 1) is true when P(k) is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.

Example 3 For all $n \geq 1$, prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Solution We can write

$$\text{P}(n): \quad \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

We note that P(1): $\frac{1}{1.2}=\frac{1}{2}=\frac{1}{1+1}$, which is true. Thus, P(n) is true for n = 1.

Assume that P(k) is true for some natural number k,

i.e., $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \text{... (1)}$$

We need to prove that P(k + 1) is true whenever P(k) is true. We have

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)\,(k+2)}$$

$$=\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \text{[Using (1)]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k^2+2k+1)}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

Thus P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

Example 4 For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Solution We can write

P(n) : $7^n - 3^n$ is divisible by 4.

We note that

P(1): $7^1 - 3^1 = 4$ which is divisible by 4. Thus P(n) is true for $n = 1$

Let P(k) be true for some natural number k,

i.e., P(k) : $7^k - 3^k$ is divisible by 4.

We can write $7^k - 3^k = 4d$, where $d \in \mathbf{N}$.

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

Now $\quad 7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$

$= 7(7^k - 3^k) + (7-3)3^k = 7(4d) + (7-3)3^k$

$= 7(4d) + 4.3^k = 4(7d + 3^k)$

From the last line, we see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4. Thus, P(k + 1) is true when P(k) is true. Therefore, by principle of mathematical induction the statement is true for every positive integer n.

Example 5 Prove that $(1 + x)^n \geq (1 + nx)$, for all natural number n, where $x > -1$.

Solution Let P(n) be the given statement,

i.e., P(n): $(1 + x)^n \geq (1 + nx)$, for $x > -1$.

We note that P(n) is true when $n = 1$, since $(1+x) \geq (1 + x)$ for $x > -1$

Assume that

P(k): $(1 + x)^k \geq (1 + kx)$, $x > -1$ is true. ... (1)

We want to prove that P(k + 1) is true for $x > -1$ whenever P(k) is true. ... (2)

Consider the identity

$(1 + x)^{k+1} = (1 + x)^k (1 + x)$

Given that $x > -1$, so $(1+x) > 0$.

Therefore , by using $(1 + x)^k \geq (1 + kx)$, we have

$(1 + x)^{k+1} \geq (1 + kx)(1 + x)$

i.e. $\quad (1 + x)^{k+1} \geq (1 + x + kx + kx^2)$. ... (3)

Here k is a natural number and $x^2 \geq 0$ so that $kx^2 \geq 0$. Therefore

$$(1 + x + kx + kx^2) \geq (1 + x + kx),$$

and so we obtain

$$(1 + x)^{k+1} \geq (1 + x + kx)$$

i.e. $$(1 + x)^{k+1} \geq [1 + (1 + k)x]$$

Thus, the statement in (2) is established. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

Example 6 Prove that

$$2.7^n + 3.5^n - 5 \text{ is divisible by 24, for all } n \in \mathbf{N}.$$

Solution Let the statement P(n) be defined as

$$P(n): \; 2.7^n + 3.5^n - 5 \text{ is divisible by 24.}$$

We note that P(n) is true for $n = 1$, since $2.7 + 3.5 - 5 = 24$, which is divisible by 24.

Assume that P(k) is true

i.e. $$2.7^k + 3.5^k - 5 = 24q, \text{ when } q \in \mathbf{N} \qquad \text{... (1)}$$

Now, we wish to prove that P(k + 1) is true whenever P(k) is true.

We have

$$2.7^{k+1} + 3.5^{k+1} - 5 = 2.7^k \,.\, 7^1 + 3.5^k \,.\, 5^1 - 5$$

$$= 7\,[2.7^k + 3.5^k - 5 - 3.5^k + 5] + 3.5^k \,.\, 5 - 5$$

$$= 7\,[24q - 3.5^k + 5] + 15.5^k - 5$$

$$= 7 \times 24q - 21.5^k + 35 + 15.5^k - 5$$

$$= 7 \times 24q - 6.5^k + 30$$

$$= 7 \times 24q - 6\,(5^k - 5)$$

$$= 7 \times 24q - 6\,(4p) \; [(5^k - 5) \text{ is a multiple of 4 (why?)}]$$

$$= 7 \times 24q - 24p$$

$$= 24\,(7q - p)$$

$$= 24 \times r;\; r = 7q - p, \text{ is some natural number.} \qquad \text{... (2)}$$

The expression on the R.H.S. of (1) is divisible by 24. Thus P(k + 1) is true whenever P(k) is true.

Hence, by principle of mathematical induction, P(n) is true for all $n \in$ N.

Example 7 Prove that

$$1^2 + 2^2 + ... + n^2 > \frac{n^3}{3}, n \in \mathbf{N}$$

Solution Let P(n) be the given statement.

i.e., $P(n) : 1^2 + 2^2 + ... + n^2 > \frac{n^3}{3}, \ n \in \mathbf{N}$

We note that P(n) is true for $n = 1$ since $1^2 > \frac{1^3}{3}$

Assume that P(k) is true

i.e. $$P(k) : 1^2 + 2^2 + ... + k^2 > \frac{k^3}{3} \quad ...(1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

We have $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2$

$$= (1^2 + 2^2 + ... + k^2) + (k+1)^2 > \frac{k^3}{3} + (k+1)^2 \quad \text{[by (1)]}$$

$$= \frac{1}{3}[k^3 + 3k^2 + 6k + 3]$$

$$= \frac{1}{3}[(k + 1)^3 + 3k + 2] > \frac{1}{3}(k + 1)^3$$

Therefore, P(k + 1) is also true whenever P(k) is true. Hence, by mathematical induction P(n) is true for all $n \in \mathbf{N}$.

Example 8 Prove the rule of exponents $(ab)^n = a^n b^n$
by using principle of mathematical induction for every natural number.

Solution Let P(n) be the given statement
i.e. $P(n) : (ab)^n = a^n b^n$.

We note that P(n) is true for $n = 1$ since $(ab)^1 = a^1 b^1$.

Let P(k) be true, i.e.,

$$(ab)^k = a^k b^k \quad ... (1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

Now, we have

$$(ab)^{k+1} = (ab)^k (ab)$$

$$= (a^k\, b^k)\,(ab) \qquad\qquad [\text{by } (1)]$$

$$= (a^k \,.\, a^1)\,(b^k \,.\, b^1) = a^{k+1} \,.\, b^{k+1}$$

Therefore, P(k + 1) is also true whenever P(k) is true. Hence, by principle of mathematical induction, P(n) is true for all $n \in$ N.

EXERCISE 4.1

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

1. $1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{(3^n - 1)}{2}$.

2. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

3. $1 + \dfrac{1}{(1+2)} + \dfrac{1}{(1+2+3)} + ... + \dfrac{1}{(1+2+3+...n)} = \dfrac{2n}{(n+1)}$.

4. $1.2.3 + 2.3.4 + \ldots + n(n+1)\,(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

5. $1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \dfrac{(2n-1)3^{n+1} + 3}{4}$.

6. $1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \left[\dfrac{n(n+1)(n+2)}{3}\right]$.

7. $1.3 + 3.5 + 5.7 + \ldots + (2n-1)\,(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$.

8. $1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n-1)\, 2^{n+1} + 2$.

9. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$.

10. $\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + ... + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{(6n+4)}$.

11. $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + ... + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$.

12. $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$.

13. $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)\ldots\left(1+\dfrac{(2n+1)}{n^2}\right) = (n+1)^2$.

14. $\left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\ldots\left(1+\dfrac{1}{n}\right) = (n+1)$.

15. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

16. $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+\ldots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{(3n+1)}$.

17. $\dfrac{1}{3.5}+\dfrac{1}{5.7}+\dfrac{1}{7.9}+\ldots+\dfrac{1}{(2n+1)(2n+3)}=\dfrac{n}{3(2n+3)}$.

18. $1 + 2 + 3 + \ldots + n < \dfrac{1}{8}(2n+1)^2$.

19. $n(n+1)(n+5)$ is a multiple of 3.

20. $10^{2n-1} + 1$ is divisible by 11.

21. $x^{2n} - y^{2n}$ is divisible by $x + y$.

22. $3^{2n+2} - 8n - 9$ is divisible by 8.

23. $41^n - 14^n$ is a multiple of 27.

24. $(2n + 7) < (n + 3)^2$.

Summary

- One key basis for mathematical thinking is deductive reasoning. In contrast to deduction, inductive reasoning depends on working with different cases and developing a conjecture by observing incidences till we have observed each and every case. Thus, in simple language we can say the word 'induction' means the generalisation from particular cases or facts.
- The principle of mathematical induction is one such tool which can be used to prove a wide variety of mathematical statements. Each such statement is assumed as P(n) associated with positive integer n, for which the correctness

for the case $n = 1$ is examined. Then assuming the truth of P(k) for some positive integer k, the truth of P (k+1) is established.

Historical Note

Unlike other concepts and methods, proof by mathematical induction is not the invention of a particular individual at a fixed moment. It is said that the principle of mathematical induction was known by the Pythagoreans.

The French mathematician Blaise Pascal is credited with the origin of the principle of mathematical induction.

The name induction was used by the English mathematician John Wallis.

Later the principle was employed to provide a proof of the binomial theorem.

De Morgan contributed many accomplishments in the field of mathematics on many different subjects. He was the first person to define and name "mathematical induction" and developed De Morgan's rule to determine the convergence of a mathematical series.

G. Peano undertook the task of deducing the properties of natural numbers from a set of explicitly stated assumptions, now known as Peano's axioms.The principle of mathematical induction is a restatement of one of the Peano's axioms.

— ❖ —

Chapter4:

Complex Numbers and Quadratic Equations

Polar representation of complex numbers. Statement of Fundamental Theorem of Algebra, solution of quadratic equations (with real coefficients) in the complex number system.

5.5.1 ***Polar representation of a complex number*** Let the point P represent the non-zero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis (Fig 5.4).

We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the x axis as the initial line.

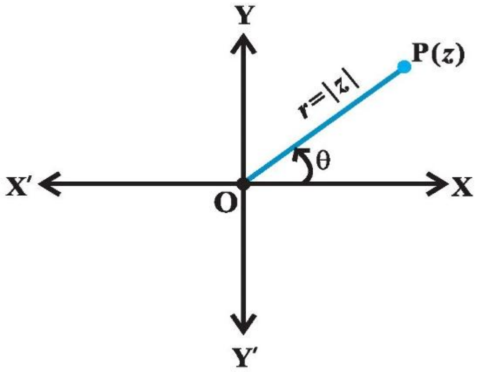

Fig 5.4

We have, $x = r \cos \theta$, $y = r \sin \theta$ and therefore, $z = r (\cos \theta + i \sin \theta)$. The latter is said to be the *polar form of the complex number*. Here $r = \sqrt{x^2 + y^2} = |z|$ is the modulus of z and θ is called the argument (or amplitude) of z which is denoted by arg z.

For any complex number $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be such an interval. We shall take the value of θ such that $-\pi < \theta \leq \pi$, called **principal argument** of z and is denoted by arg z, unless specified otherwise. (Figs. 5.5 and 5.6)

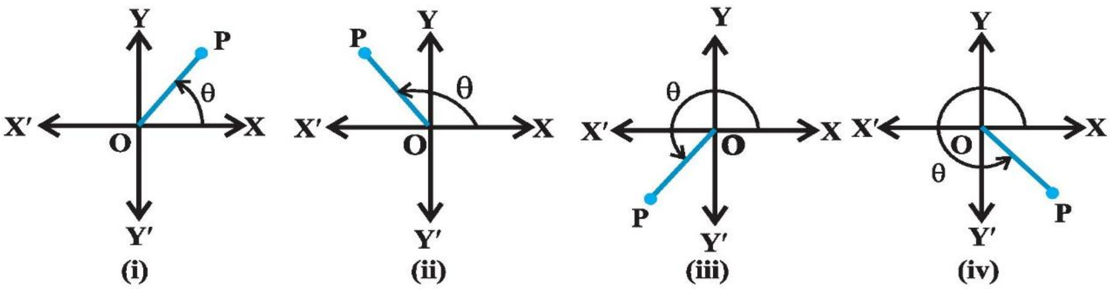

Fig 5.5 $(0 \leq \theta < 2\pi)$

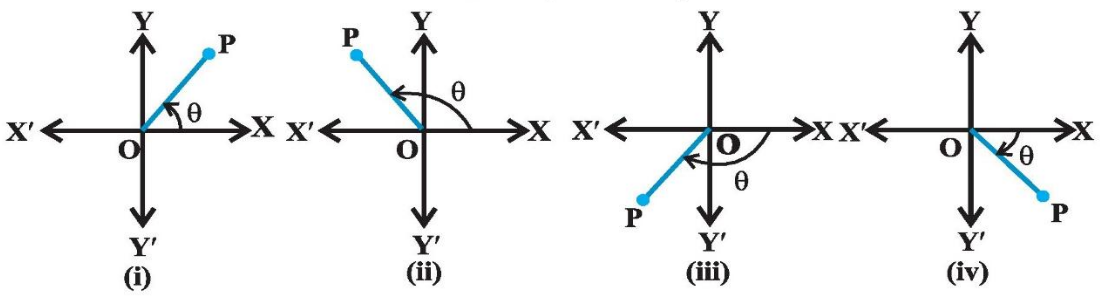

Fig 5.6 $(-\pi < \theta \leq \pi)$

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Example 7 Represent the complex number $z = 1 + i\sqrt{3}$ in the polar form.

Solution Let $1 = r\cos\theta$, $\sqrt{3} = r\sin\theta$

By squaring and adding, we get

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 4$$

i.e., $r = \sqrt{4} = 2$ (conventionally, $r > 0$)

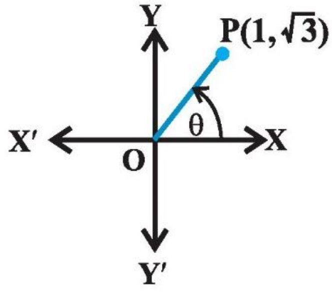

Fig 5.7

Therefore, $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, which gives $\theta = \frac{\pi}{3}$

Therefore, required polar form is $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

The complex number $z = 1 + i\sqrt{3}$ is represented as shown in Fig 5.7.

Example 8 Convert the complex number $\frac{-16}{1 + i\sqrt{3}}$ into polar form.

Solution The given complex number $\frac{-16}{1 + i\sqrt{3}} = \frac{-16}{1 + i\sqrt{3}} \times \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}}$

$$= \frac{-16\left(1 - i\sqrt{3}\right)}{1 - \left(i\sqrt{3}\right)^2} = \frac{-16\left(1 - i\sqrt{3}\right)}{1 + 3} = -4\left(1 - i\sqrt{3}\right) = -4 + i4\sqrt{3} \text{ (Fig 5.8).}$$

Let $-4 = r\cos\theta$, $4\sqrt{3} = r\sin\theta$

By squaring and adding, we get

$$16 + 48 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

which gives $r^2 = 64$, i.e., $r = 8$

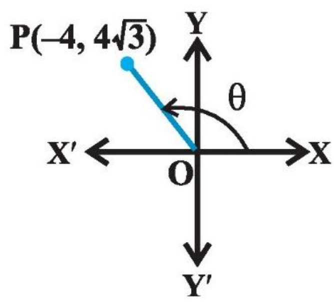

Fig 5.8

Hence $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Thus, the required polar form is $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

EXERCISE 5.2

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{3}$ **2.** $z = -\sqrt{3} + i$

Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:

3. $1 - i$ **4.** $-1 + i$ **5.** $-1 - i$

6. -3 **7.** $\sqrt{3} + i$ **8.** i

5.6 Quadratic Equations

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminant is non-negative, i.e., ≥ 0,

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$.

Also, let us assume that the $b^2 - 4ac < 0$.

Now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

> **Note** At this point of time, some would be interested to know as to how many roots does an equation have? In this regard, the following theorem known as the *Fundamental theorem of Algebra* is stated below (without proof).

"A polynomial equation has at least one root."

As a consequence of this theorem, the following result, which is of immense importance, is arrived at:

"A polynomial equation of degree n has n roots."

Example 9 Solve $x^2 + 2 = 0$

Solution We have, $x^2 + 2 = 0$

or $x^2 = -2$ i.e., $x = \pm\sqrt{-2} = \pm\sqrt{2}\, i$

Example 10 Solve $x^2 + x + 1 = 0$

Solution Here, $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Therefore, the solutions are given by $x = \dfrac{-1 \pm \sqrt{-3}}{2 \times 1} = \dfrac{-1 \pm \sqrt{3}i}{2}$

Example 11 Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$

Solution Here, the discriminant of the equation is

$$1^2 - 4 \times \sqrt{5} \times \sqrt{5} = 1 - 20 = -19$$

Therefore, the solutions are

$$\frac{-1 \pm \sqrt{-19}}{2\sqrt{5}} = \frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}.$$

EXERCISE 5.3

Solve each of the following equations:

1. $x^2 + 3 = 0$ **2.** $2x^2 + x + 1 = 0$ **3.** $x^2 + 3x + 9 = 0$

4. $-x^2 + x - 2 = 0$ **5.** $x^2 + 3x + 5 = 0$ **6.** $x^2 - x + 2 = 0$

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$ **8.** $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

9. $x^2 + x + \dfrac{1}{\sqrt{2}} = 0$ **10.** $x^2 + \dfrac{x}{\sqrt{2}} + 1 = 0$

Chapter 5: Linear Inequalities

Graphical solution of linear inequalities in two variables. Graphical method of finding a solution of system of linear inequalities in two variables

6.4 Graphical Solution of Linear Inequalities in Two Variables

In earlier section, we have seen that a graph of an inequality in one variable is a visual representation and is a convenient way to represent the solutions of the inequality. Now, we will discuss graph of a linear inequality in two variables.

We know that a line divides the Cartesian plane into two parts. Each part is known as a half plane. A vertical line will divide the plane in left and right half planes and a non-vertical line will divide the plane into lower and upper half planes (Figs. 6.3 and 6.4).

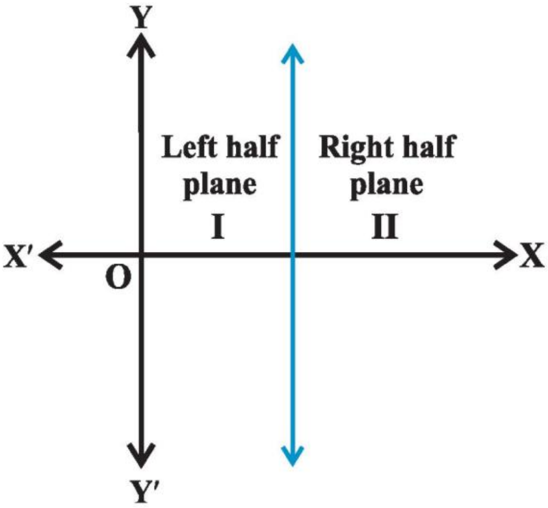

Fig 6.3

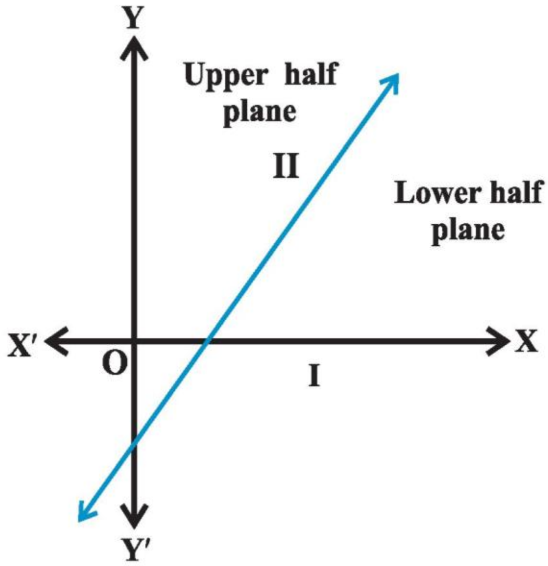

Fig 6.4

A point in the Cartesian plane will either lie on a line or will lie in either of the half planes I or II. We shall now examine the relationship, if any, of the points in the plane and the inequalities $ax + by < c$ or $ax + by > c$.

Let us consider the line

$$ax + by = c, \quad a \neq 0, \quad b \neq 0 \qquad \dots (1)$$

`

There are three possibilities namely:

(i) $ax + by = c$ (ii) $ax + by > c$ (iii) $ax + by < c$.

In case (i), clearly, all points (x, y) satisfying (i) lie on the line it represents and conversely. Consider case (ii), let us first assume that $b > 0$. Consider a point P (α,β) on the line $ax + by = c$, $b > 0$, so that $a\alpha + b\beta = c$. Take an arbitrary point Q (α, γ) in the half plane II (Fig 6.5).

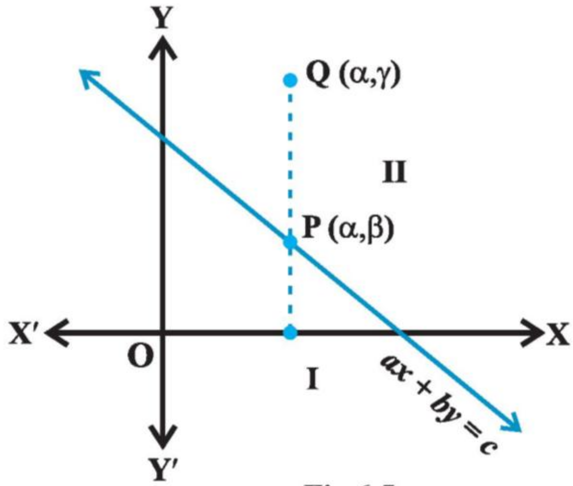

Fig 6.5

Now, from Fig 6.5, we interpret,

$\gamma > \beta$ (Why?)

or $b\gamma > b\beta$ or $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

or $a\alpha + b\gamma > c$

i.e., Q(α, γ) satisfies the inequality $ax + by > c$.

Thus, all the points lying in the half plane II above the line $ax + by = c$ satisfies the inequality $ax + by > c$. Conversely, let (α, β) be a point on line $ax + by = c$ and an arbitrary point Q(α, γ) satisfying

$ax + by > c$

so that $a\alpha + b\gamma > c$

$\Rightarrow$ $a\alpha + b\gamma > a\alpha + b\beta$ (Why?)

$\Rightarrow$ $\gamma > \beta$ (as $b > 0$)

This means that the point (α, γ) lies in the half plane II.

Thus, any point in the half plane II satisfies $ax + by > c$, and conversely any point satisfying the inequality $ax + by > c$ lies in half plane II.

In case $b < 0$, we can similarly prove that any point satisfying $ax + by > c$ lies in the half plane I, and conversely.

Hence, we deduce that all points satisfying $ax + by > c$ lies in one of the half planes II or I according as $b > 0$ or $b < 0$, and conversely.

Thus, graph of the inequality $ax + by > c$ will be one of the half plane (called *solution region*) and represented by shading in the corresponding half plane.

Note 1 The region containing all the solutions of an inequality is called the *solution region.*

2. In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) (not on line) and check whether it satisfies the inequality or not. If it satisfies, then the inequality represents the half plane and shade the region

which contains the point, otherwise, the inequality represents that half plane which does not contain the point within it. For convenience, the point (0, 0) is preferred.

3. If an inequality is of the type $ax + by \geq c$ or $ax + by \leq c$, then the points on the line $ax + by = c$ are also included in the solution region. So draw a dark line in the solution region.

4. If an inequality is of the form $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So draw a broken or dotted line in the solution region.

In Section 6.2, we obtained the following linear inequalities in two variables *x and y*: $40x + 20y \leq 120$ ... (1)
while translating the word problem of purchasing of registers and pens by Reshma.

Let us now solve this inequality keeping in mind that x and y can be only whole numbers, since the number of articles cannot be a fraction or a negative number. In this case, we find the pairs of values of x and y, which make the statement (1) true. In fact, the set of such pairs will be the *solution set* of the inequality (1).

To start with, let $x = 0$. Then L.H.S. of (1) is

$$40x + 20y = 40\ (0) + 20y = 20y.$$

Thus, we have

$$20y \leq 120 \text{ or } y \leq 6 \qquad \text{... (2)}$$

For $x = 0$, the corresponding values of y can be 0, 1, 2, 3, 4, 5, 6 only. In this case, the solutions of (1) are (0, 0), (0, 1), (0,2), (0,3), (0,4), (0, 5) and (0, 6).

Similarly, other solutions of (1), when $x = 1$, 2 and 3 are: (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 0), (2, 1), (2, 2), (3, 0)
This is shown in Fig 6.6.

Let us now extend the domain of x and y from whole numbers to real numbers, and see what will be the solutions of (1) in this case. You will see that the graphical method of solution will be very convenient in this case. For this purpose, let us consider the (corresponding) equation and draw its graph.

$$40x + 20y = 120 \qquad \text{... (3)}$$

In order to draw the graph of the inequality (1), we take one point say (0, 0), in half plane I and check whether values of x and y satisfy the inequality or not.

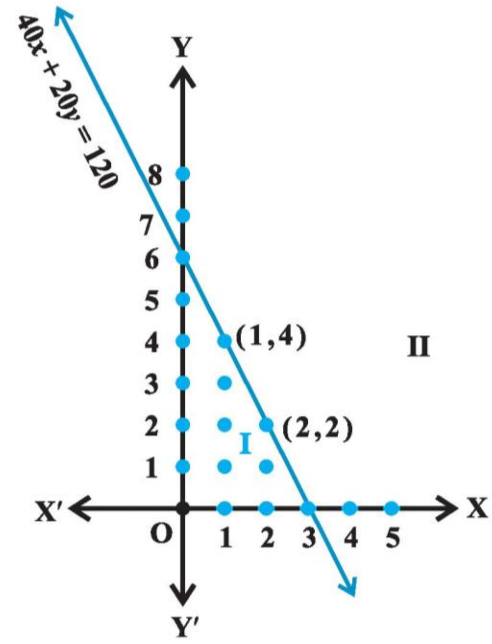

Fig 6.6

We observe that $x = 0$, $y = 0$ satisfy the inequality. Thus, we say that the half plane I is the graph (Fig 6.7) of the inequality. Since the points on the line also satisfy the inequality (1) above, the line is also a part of the graph.

Thus, the graph of the given inequality is half plane I including the line itself. Clearly half plane II is not the part of the graph. Hence, *solutions* of inequality (1) will consist of all the points of its graph (half plane I including the line).

We shall now consider some examples to explain the above procedure for solving a linear inequality involving two variables.

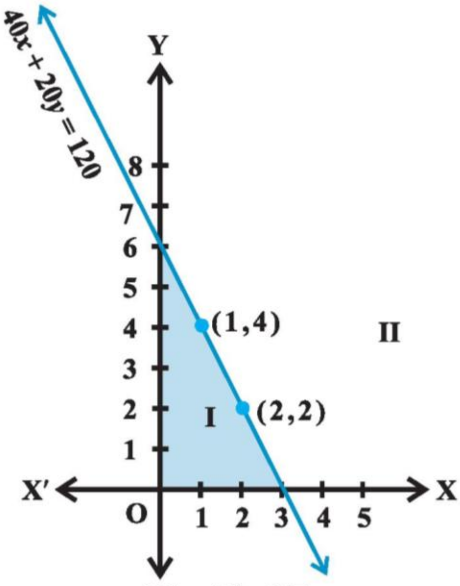

Fig 6.7

Example 9 Solve $3x + 2y > 6$ graphically.

Solution Graph of $3x + 2y = 6$ is given as dotted line in the Fig 6.8.

This line divides the xy-plane in two half planes I and II. We select a point (not on the line), say (0, 0), which lies in one of the half planes (Fig 6.8) and determine if this point satisfies the given inequality, we note that

$3\,(0) + 2\,(0) > 6$

or $0 > 6$, which is false.

Hence, half plane I is not the solution region of the given inequality. Clearly, any point on the line does not satisfy the given strict inequality. In other words, the shaded half plane II excluding the points on the line is the solution region of the inequality.

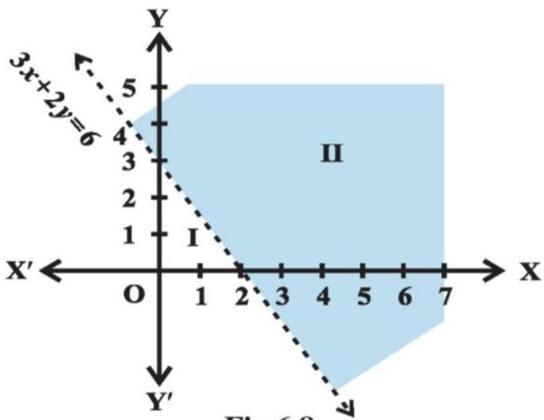

Fig 6.8

Example 10 Solve $3x - 6 \geq 0$ graphically in two dimensional plane.

Solution Graph of $3x - 6 = 0$ is given in the Fig 6.9.

We select a point, say (0, 0) and substituting it in given inequality, we see that:

$3\,(0) - 6 \geq 0$ or $-6 \geq 0$ which is false.

Thus, the solution region is the shaded region on the right hand side of the line $x = 2$.

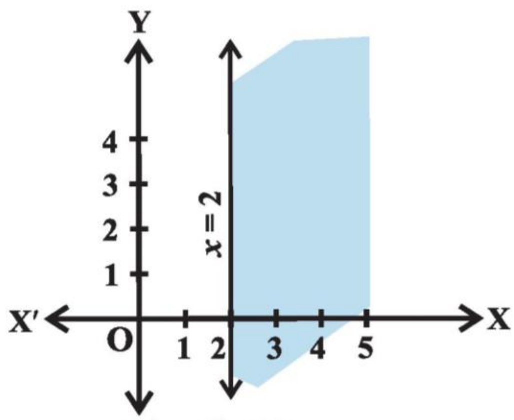

Fig 6.9

Example 11 Solve $y < 2$ graphically.

Solution Graph of $y = 2$ is given in the Fig 6.10.

Let us select a point, (0, 0) in lower half plane I and putting $y = 0$ in the given inequality, we see that

$1 \times 0 < 2$ or $0 < 2$ which is true.

Thus, the solution region is the shaded region below the line $y = 2$. Hence, every point below the line (excluding all the points on the line) determines the solution of the given inequality.

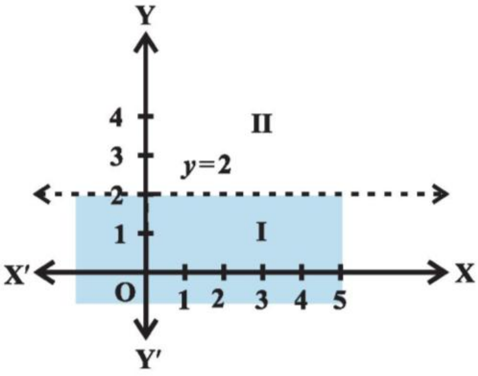

Fig 6.10

EXERCISE 6.2

Solve the following inequalities graphically in two-dimensional plane:

1. $x + y < 5$
2. $2x + y \geq 6$
3. $3x + 4y \leq 12$
4. $y + 8 \geq 2x$
5. $x - y \leq 2$
6. $2x - 3y > 6$
7. $-3x + 2y \geq -6$
8. $3y - 5x < 30$
9. $y < -2$
10. $x > -3$.

6.5 Solution of System of Linear Inequalities in Two Variables

In previous Section, you have learnt how to solve linear inequality in one or two variables graphically. We will now illustrate the method for solving a system of linear inequalities in two variables graphically through some examples.

Example 12 Solve the following system of linear inequalities graphically.

$$x + y \geq 5 \quad \text{... (1)}$$

$$x - y \leq 3 \quad \text{... (2)}$$

Solution The graph of linear equation

$$x + y = 5$$

is drawn in Fig 6.11.

We note that solution of inequality (1) is represented by the shaded region above the line $x + y = 5$, including the points on the line.

On the same set of axes, we draw the graph of the equation $x - y = 3$ as shown in Fig 6.11. Then we note that inequality (2) represents the shaded region above

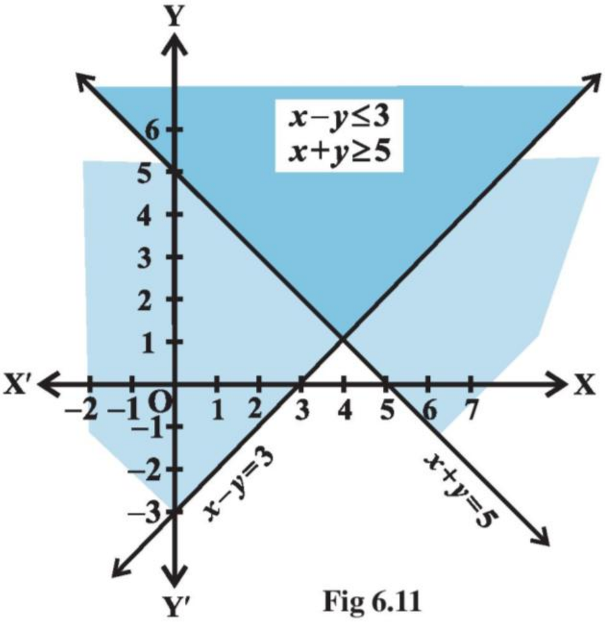

Fig 6.11

the line $x - y = 3$, including the points on the line.

Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities.

Example 13 Solve the following system of inequalities graphically

$5x + 4y \leq 40$ ... (1)

$x \geq 2$ ... (2)

$y \geq 3$ ... (3)

Solution We first draw the graph of the line

$5x + 4y = 40$, $x = 2$ and $y = 3$

Then we note that the inequality (1) represents shaded region below the line $5x + 4y = 40$ and inequality (2) represents the shaded region right of line $x = 2$ but inequality (3) represents the shaded region above the line $y = 3$. Hence, shaded region (Fig 6.12) including all the point on the lines are also the solution of the given system of the linear inequalities.

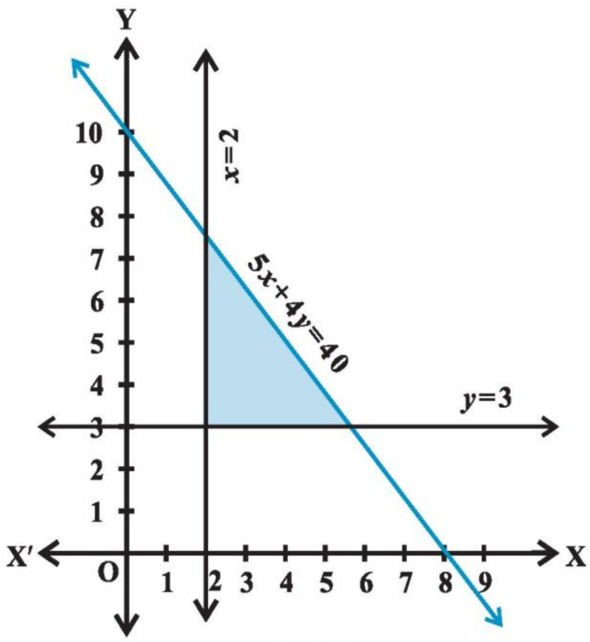

Fig 6.12

In many practical situations involving system of inequalities the variable x and y often represent quantities that cannot have negative values, for example, number of units produced, number of articles purchased, number of hours worked, etc. Clearly, in such cases, $x \geq 0$, $y \geq 0$ and the solution region lies only in the first quadrant.

Example 14 Solve the following system of inequalities

$8x + 3y \leq 100$ ... (1)

$x \geq 0$ ... (2)

$y \geq 0$ ... (3)

Solution We draw the graph of the line

$8x + 3y = 100$

The inequality $8x + 3y \leq 100$ represents the shaded region below the line, including the points on the line $8x + 3y = 100$ (Fig 6.13).

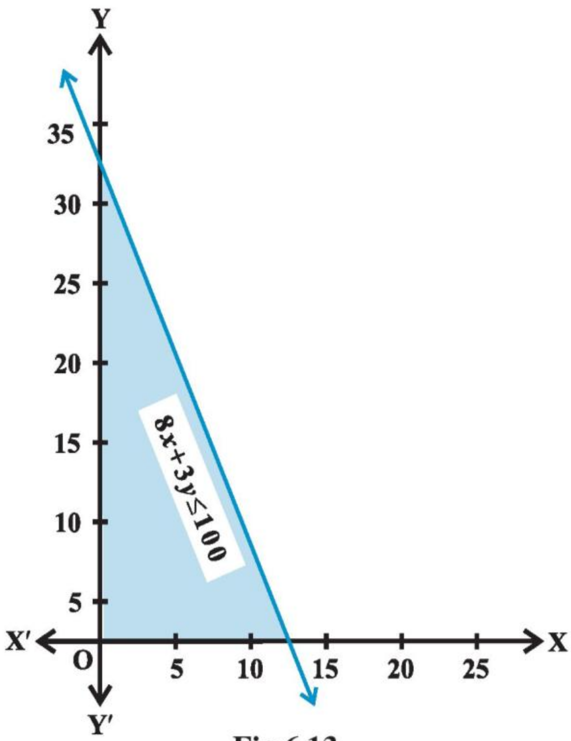

Fig 6.13

Since $x \geq 0$, $y \geq 0$, every point in the shaded region in the first quadrant, including the points on the line and the axes, represents the solution of the given system of inequalities.

Example 15 Solve the following system of inequalities graphically

$x + 2y \leq 8$ ... (1)

$2x + y \leq 8$ ... (2)

$x \geq 0$ ... (3)

$y \geq 0$ ... (4)

Solution We draw the graphs of the lines $x + 2y = 8$ and $2x + y = 8$. The inequality (1) and (2) represent the region below the two lines, including the point on the respective lines.

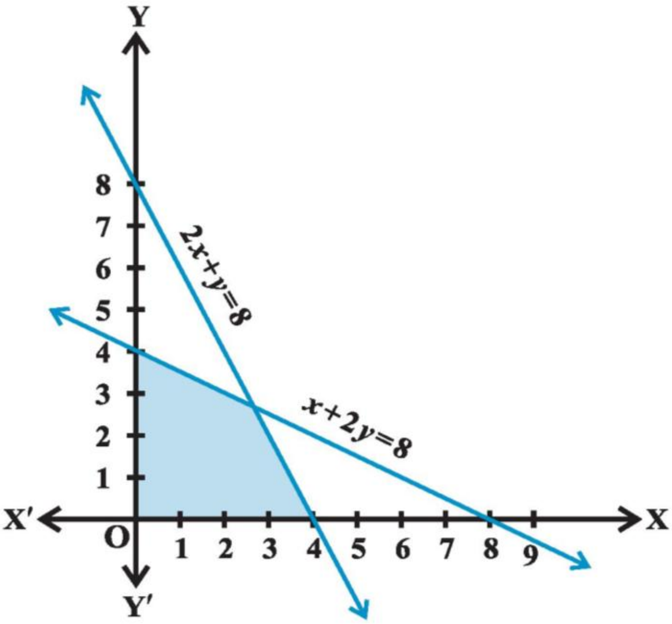

Fig 6.14

Since $x \geq 0$, $y \geq 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequalities (Fig 6.14).

EXERCISE 6.3

Solve the following system of inequalities graphically:

1. $x \geq 3, y \geq 2$
2. $3x + 2y \leq 12,\ x \geq 1, y \geq 2$
3. $2x + y \geq 6, 3x + 4y \leq 12$
4. $x + y \geq 4,\ 2x - y < 0$
5. $2x - y > 1, x - 2y < -1$
6. $x + y \leq 6,\ x + y \geq 4$
7. $2x + y \geq 8,\ x + 2y \geq 10$
8. $x + y \leq 9,\ y > x,\ x \geq 0$
9. $5x + 4y \leq 20,\ x \geq 1, y \geq 2$
10. $3x + 4y \leq 60, x + 3y \leq 30, x \geq 0,\ y \geq 0$
11. $2x + y \geq 4,\ x + y \leq 3,\ 2x - 3y \leq 6$
12. $x - 2y \leq 3, 3x + 4y \geq 12,\ x \geq 0, y \geq 1$
13. $4x + 3y \leq 60, y \geq 2x,\ x \geq 3,\ x, y \geq 0$
14. $3x + 2y \leq 150, x + 4y \leq 80,\ x \leq 15,\ y \geq 0, x \geq 0$
15. $x + 2y \leq 10, x + y \geq 1, x - y \leq 0, x \geq 0, y \geq 0$

Chapter 7: Binomial Theorem

General and middle term in binomial expansion.

8.3 General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0 a^n$, the second term is ${}^nC_1 a^{n-1}b$, the third term is ${}^nC_2 a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_r a^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.

2. Regarding the middle term in the expansion $(a + b)^n$, we have

(i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term.

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

(ii) If n is odd, then $n + 1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion

$(2x - y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4^{th} and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5^{th} term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., $(n + 1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17^{th} and 18^{th} terms of the expansion $(2 + a)^{50}$ are equal.

Solution The $(r + 1)^{th}$ term of the expansion $(x + y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17^{th} term, we have, $r + 1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16} (2)^{50-16} a^{16}$

$= {}^{50}C_{16}\, 2^{34}\, a^{16}.$

Similarly, $T_{18} = {}^{50}C_{17}\, 2^{33}\, a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16} (2)^{34} a^{16} = {}^{50}C_{17} (2)^{33} a^{17}$

Therefore $$\frac{{}^{50}C_{16} . 2^{34}}{{}^{50}C_{17} . 2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\;.\;33!}{50!} \times 2 = 1$$

Example 6 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!} 2n\, x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{\text{th}}$,

i.e., $(n + 1)^{\text{th}}$ term which is given by,

$$T_{n+1} = {}^{2n}C_n (1)^{2n-n}(x)^n = {}^{2n}C_n x^n = \frac{(2n)!}{n!\ n!} x^n$$

$$= \frac{2n\,(2n-1)\ (2n-2)\ \ldots 4.3.2.1}{n!\ n!} x^n$$

$$= \frac{1.2.3.4\ldots(2n-2)(2n-1)(2n)}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)][2.4.6\ldots(2n)]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]2^n\,[1.2.3\ldots n]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]\,n!}{n!\ n!} 2^n . x^n$$

$$= \frac{1.3.5\ldots(2n-1)}{n!} 2^n\ x^n$$

Example 7 Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Solution Suppose x^6y^3 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(x + 2y)^9$.

Now $\qquad T_{r+1} = {}^9C_r\, x^{9-r}\,(2y)^r = {}^9C_r\, 2^r . x^{9-r} . y^r$.

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}, we get $r = 3$.

Thus, the coefficient of x^6y^3 is

$${}^9C_3\, 2^3 = \frac{9!}{3!\,6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672.$$

Example 8 The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Solution Given that second term $T_2 = 240$

We have $T_2 = {}^nC_1 x^{n-1} . a$

So ${}^nC_1 x^{n-1} . a = 240$ ... (1)

Similarly ${}^nC_2 x^{n-2} a^2 = 720$ ... (2)

and ${}^nC_3 x^{n-3} a^3 = 1080$ ... (3)

Dividing (2) by (1), we get

$$\frac{{}^nC_2 x^{n-2} a^2}{{}^nC_1 x^{n-1} a} = \frac{720}{240} \quad \text{i.e.,} \quad \frac{(n-1)!}{(n-2)!} . \frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{(n-1)} \qquad \text{... (4)}$$

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)} \qquad \text{... (5)}$$

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)} . \qquad \text{Thus, } n = 5$$

Hence, from (1), $5x^4 a = 240$, and from (4), $\frac{a}{x} = \frac{3}{2}$

Solving these equations for a and x, we get $x = 2$ and $a = 3$.

Example 9 The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio1: 7 : 42. Find n.

Solution Suppose the three consecutive terms in the expansion of $(1 + a)^n$ are $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms.

The $(r - 1)^{th}$ term is ${}^nC_{r-2} a^{r-2}$, and its coefficient is ${}^nC_{r-2}$. Similarly, the coefficients of r^{th} and $(r + 1)^{th}$ terms are ${}^nC_{r-1}$ and ${}^nC_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0 \qquad \text{... (1)}$$

and $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42}, \text{ i.e., } n - 7r + 1 = 0 \qquad \text{... (2)}$$

Solving equations(1) and (2), we get, $n = 55$.

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$

2. a^5b^7 in $(a - 2b)^{12}$.

Write the general term in the expansion of

3. $(x^2 - y)^6$

4. $(x^2 - yx)^{12}, x \neq 0$.

5. Find the 4^{th} term in the expansion of $(x - 2y)^{12}$.

6. Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$

8. $\left(\frac{x}{3} + 9y\right)^{10}$.

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

10. The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n - 1}$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Chapter 8: Sequences and Series

Formulae for the following special sums

$$\sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2, \sum_{k=1}^{n} k^3$$

9.7 Sum to *n* Terms of Special Series

We shall now find the sum of first n terms of some special series, namely;

(i) $1 + 2 + 3 + \ldots + n$ (sum of first n natural numbers)

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2$(sum of squares of the first n natural numbers)

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3$(sum of cubes of the first n natural numbers).

Let us take them one by one.

(i) $S_n = 1 + 2 + 3 + \ldots + n$, then $S_n = \dfrac{n(n+1)}{2}$ (See Section 9.4)

(ii) Here $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

We consider the identity $k^3 - (k-1)^3 = 3k^2 - 3k + 1$

Putting $k = 1, 2\ldots, n$ successively, we obtain

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$

$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$

$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$

....

....

....

$$n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$$

Adding both sides, we get

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + n$$

By (i), we know that $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

Hence $S_n = \sum_{k=1}^{n} k^2 = \dfrac{1}{3}\left[n^3 + \dfrac{3n(n+1)}{2} - n\right] = \dfrac{1}{6}(2n^3 + 3n^2 + n)$

$$= \frac{n(n+1)(2n+1)}{6}$$

(iii) Here $S_n = 1^3 + 2^3 + \ldots + n^3$

We consider the identity, $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

Putting $k = 1, 2, 3\ldots n$, we get

$$2^4 - 1^4 = 4(1)^3 + 6(1)^2 + 4(1) + 1$$
$$3^4 - 2^4 = 4(2)^3 + 6(2)^2 + 4(2) + 1$$
$$4^4 - 3^4 = 4(3)^3 + 6(3)^2 + 4(3) + 1$$

....

....

....

$$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding both sides, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + 3^3 + ... + n^3) + 6(1^2 + 2^2 + 3^2 + ... + n^2) + 4(1 + 2 + 3 + ... + n) + n$$

$$= 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \qquad \text{... (1)}$$

From parts (i) and (ii), we know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting these values in equation (1), we obtain

$$4\sum_{k=1}^{n} k^3 = n^4 + 4n^3 + 6n^2 + 4n - \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} - n$$

or $\quad 4S_n = n^4 + 4n^3 + 6n^2 + 4n - n(2n^2 + 3n + 1) - 2n(n+1) - n$

$$= n^4 + 2n^3 + n^2$$
$$= n^2(n+1)^2.$$

Hence, $\quad S_n = \dfrac{n^2(n+1)^2}{4} = \dfrac{[n(n+1)]^2}{4}$

Example 19 Find the sum to n terms of the series**:** 5 + 11 + 19 + 29 + 41…

Solution Let us write

$$S_n = 5 + 11 + 19 + 29 + ... + a_{n-1} + a_n$$

or $$S_n = \quad 5 + 11 + 19 + ... + a_{n-2} + a_{n-1} + a_n$$

On subtraction, we get

$$0 = 5 + [6 + 8 + 10 + 12 + ...(n-1) \text{ terms}] - a_n$$

or $$a_n = 5 + \frac{(n-1)[12+(n-2)\times 2]}{2}$$

$$= 5 + (n-1)(n+4) = n^2 + 3n + 1$$

Hence $$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2+3k+1) = \sum_{k=1}^{n} k^2 + 3\sum_{1}^{n} k + n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}.$$

Example 20 Find the sum to n terms of the series whose n^{th} term is $n(n+3)$.

Solution Given that $a_n = n(n+3) = n^2 + 3n$

Thus, the sum to n terms is given by

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} = \frac{n(n+1)(n+5)}{3}.$$

EXERCISE 9.4

Find the sum to n terms of each of the series in Exercises 1 to 7.

1. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

2. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ...$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ...$

4. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ...$

5. $5^2 + 6^2 + 7^2 + ... + 20^2$

6. $3 \times 8 + 6 \times 11 + 9 \times 14 + ...$

7. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$

Find the sum to n terms of the series in Exercises 8 to 10 whose n^{th} terms is given by

8. $n(n+1)(n+4)$.

9. $n^2 + 2^n$

10. $(2n-1)^2$

Chapter 9: Straight Lines

Normal form. General equation of a line.

10.3.6 *Normal form* Suppose a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

(ii) Angle which normal makes with the positive direction of x-axis.

Let L be the line, whose perpendicular distance from origin O be OA = p and the angle between the positive x-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Fig 10.17. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the x-axis in each case.

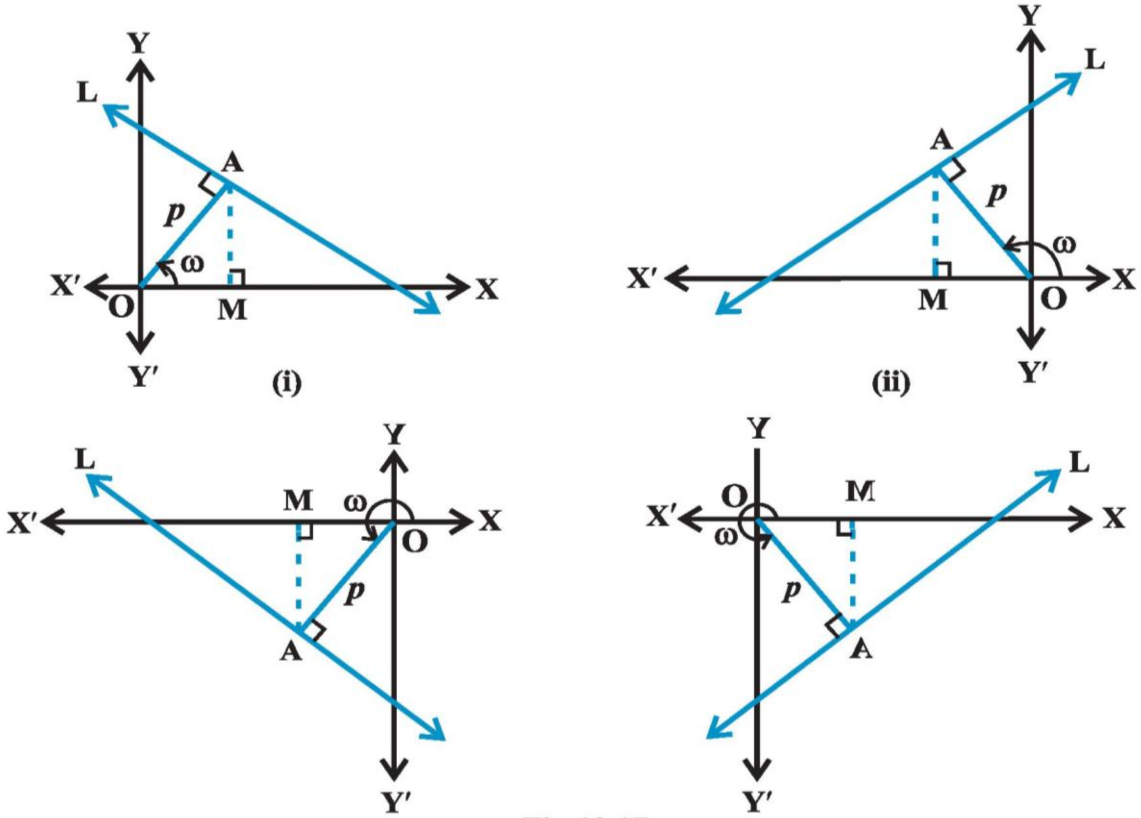

Fig 10.17

In each case, we have OM = $p \cos \omega$ and MA = $p \sin \omega$, so that the coordinates of the point A are ($p \cos \omega$, $p \sin \omega$).

Further, line L is perpendicular to OA. Therefore

$$\text{The slope of the line L} = -\frac{1}{\text{slope of OA}} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Thus, the line L has slope $-\dfrac{\cos\omega}{\sin \omega}$ and point $A\left(p\cos\omega, p\sin\omega\right)$ on it. Therefore, by

point-slope form, the equation of the line L is

`

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p(\sin^2\omega + \cos^2\omega)$$

or $$x\cos\omega + y\sin\omega = p.$$

Hence, the equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x\cos\omega + y\sin\omega = p \qquad \text{... (6)}$$

Example 11 Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

Solution Here, we are given $p = 4$ and $\omega = 15^0$ (Fig10.18).

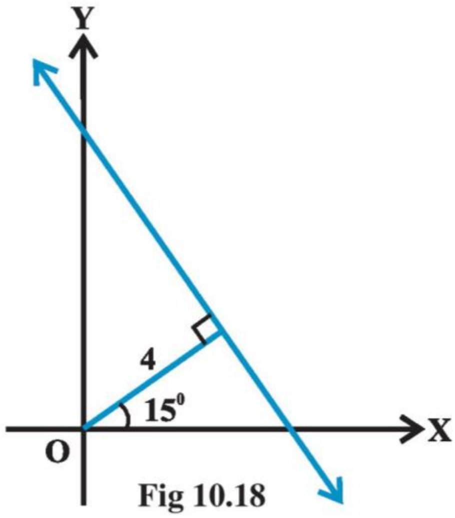

Fig 10.18

Now $$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

and $$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{(Why?)}$$

By the normal form (6) above, the equation of the line is

$$x\cos 15^0 + y\sin 15^0 = 4 \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4 \quad \text{or} \quad \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

This is the required equation.

Example 12 The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.

Solution Assuming F along x-axis and K along y-axis, we have two points (32, 273) and (212, 373) in XY-plane. By two-point form, the point (F, K) satisfies the equation

$$K - 273 = \frac{373-273}{212-32}(F-32) \quad \text{or} \quad K - 273 = \frac{100}{180}(F-32)$$

or $$K = \frac{5}{9}(F-32) + 273 \qquad \text{... (1)}$$

which is the required relation.

When K = 0, Equation (1) gives

$$0=\frac{5}{9}(F-32)+273 \quad \text{or} \quad F-32=-\frac{273\times 9}{5}=-\ 491.4 \quad \text{or} \quad F=\ -459.4\,.$$

Alternate method We know that simplest form of the equation of a line is $y = mx + c$. Again assuming F along *x-axis* and K along *y-axis*, we can take equation in the form

$$K = mF + c \qquad \text{... (1)}$$

Equation (1) is satisfied by (32, 273) and (212, 373). Therefore

$$273 = 32m + c \qquad \text{... (2)}$$

and

$$373 = 212m + c \qquad \text{... (3)}$$

Solving (2) and (3), we get

$$m = \frac{5}{9} \text{ and } c = \frac{2297}{9}.$$

Putting the values of m and c in (1), we get

$$K=\frac{5}{9}F+\frac{2297}{9} \qquad \text{... (4)}$$

which is the required relation. When K = 0, (4) gives F = – 459.4.

> **Note** We know, that the equation $y = mx + c$, contains two constants, namely, m and c. For finding these two constants, we need two conditions satisfied by the equation of line. In all the examples above, we are given two conditions to determine the equation of the line.

10.4 General Equation of a Line

In earlier classes, we have studied general equation of first degree in two variables, Ax + By + C = 0, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation Ax + By + C = 0 is always a straight line.

Therefore, any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

10.4.1 ***Different forms of*** **Ax + By + *C* = 0** The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

(a) ***Slope-intercept form*** If B ≠ 0, then Ax + By + C = 0 can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ or } y = mx + c \qquad \text{... (1)}$$

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and y-intercept is $-\frac{C}{B}$.

If B = 0, then $x = -\frac{C}{A}$, which is a vertical line whose slope is undefined and x-intercept is $-\frac{C}{A}$.

(b) Intercept form If C ≠ 0, then Ax + By + C = 0 can be written as

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \text{ or } \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (2)}$$

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$.

We know that equation (2) is intercept form of the equation of a line whose x-intercept is $-\frac{C}{A}$ and y-intercept is $-\frac{C}{B}$.

If C = 0, then Ax + By + C = 0 can be written as Ax + By = 0, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

(c) Normal form Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation Ax + By + C = 0 or Ax + By = – C. Thus, both the equations are same and therefore, $\frac{A}{\cos \omega} = \frac{B}{\sin \omega} = -\frac{C}{p}$

which gives $$\cos\omega = -\frac{Ap}{C} \text{ and } \sin\omega = -\frac{Bp}{C}.$$

Now $$\sin^2\omega + \cos^2\omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $$p^2 = \frac{C^2}{A^2 + B^2} \quad \text{or} \quad p = \pm\frac{C}{\sqrt{A^2 + B^2}}$$

Therefore $$\cos\omega = \pm\frac{A}{\sqrt{A^2 + B^2}} \text{ and } \sin\omega = \pm\frac{B}{\sqrt{A^2 + B^2}}.$$

Thus, the normal form of the equation $Ax + By + C = 0$ is

$$x \cos\omega + y \sin\omega = p,$$

where $\cos\omega = \pm\frac{A}{\sqrt{A^2 + B^2}}$, $\sin\omega = \pm\frac{B}{\sqrt{A^2 + B^2}}$ and $p = \pm\frac{C}{\sqrt{A^2 + B^2}}$.

Proper choice of signs is made so that p should be positive.

Example 13 Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) x - and y-intercepts.

Solution (i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \qquad \text{... (1)}$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y = -10 \quad \text{or} \quad \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \qquad \text{... (2)}$$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have x-intercept as $a = -\frac{10}{3}$ and y-intercept as $b = \frac{5}{2}$.

Example 14 Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

Solution Given equation is

$$\sqrt{3}x + y - 8 = 0 \qquad \text{... (1)}$$

Dividing (1) by $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \text{ or } \cos 30° x + \sin 30° y = 4 \qquad \text{... (2)}$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30°$.

Example15 Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \qquad \text{... (1)}$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \qquad \text{... (2)}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \qquad \text{...}$$

(3)

Putting the values of m_1 and m_2 in (3), we get

$$\tan \theta = \left| \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}} \right| = \left| \frac{1-3}{2\sqrt{3}} \right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30°$. Hence, angle between two lines is either 30° or 180° – 30° = 150°.

Example 16 Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and (ii) Perpendicular if $a_1a_2 - b_1b_2 = 0$.

Solution Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \qquad \text{... (1)}$$

and

$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \qquad \text{... (2)}$$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1.m_2 = -1$, which gives

$$\frac{a_1}{b_1}.\frac{a_2}{b_2} = -1 \text{ or } a_1a_2 + b_1b_2 = 0$$

Example 17 Find the equation of a line perpendicular to the line $x - 2y - 3 = 0$ and passing through the point (1, – 2).

Solution Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \qquad \text{...(1)}$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope – 2 and passing through the point (1, – 2) is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.

12.5 Section Formula

In two dimensional geometry, we have learnt how to find the coordinates of a point dividing a line segment in a given ratio internally. Now, we extend this to three dimensional geometry as follows:

Let the two given points be P(x_1, y_1, z_1) and Q (x_2, y_2, z_2). Let the point R (x, y, z) divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously PL || RN || QM and feet of these perpendiculars lie in a XY-plane. The points L, M and N will lie on a line which is the intersection of the plane containing PL, RN and QM with the XY-plane. Through the point R draw a line ST parallel to the line LM. Line ST will intersect the line LP externally at the point S and the line MQ at T, as shown in Fig 12.5.

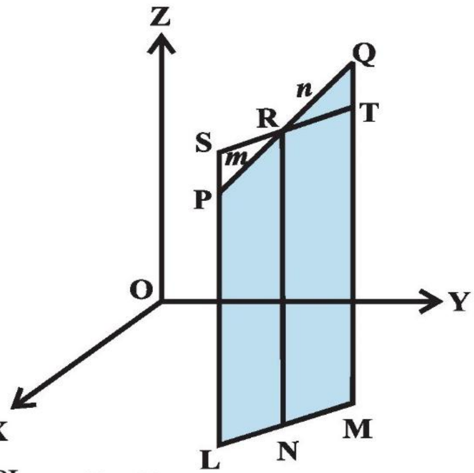

Fig 12.5

Also note that quadrilaterals LNRS and NMTR are parallelograms.

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n} = \frac{PR}{QR} = \frac{SP}{QT} = \frac{SL-PL}{QM-TM} = \frac{NR-PL}{QM-NR} = \frac{z-z_1}{z_2-z}$$

This implies $$z = \frac{mz_2 + nz_1}{m+n}$$

Similarly, by drawing perpendiculars to the XZ and YZ-planes, we get

$$y = \frac{my_2 + ny_1}{m+n} \ \ and \ \ x = \frac{mx_2 + nx_1}{m+n}$$

Hence, the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

If the point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n by $-n$ so that coordinates of point R will be

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Case 1 Coordinates of the mid-point: In case R is the mid-point of PQ, then

$m : n = 1 : 1$ so that $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ and $z = \frac{z_1 + z_2}{2}$.

These are the coordinates of the mid point of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Case 2 The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k = \frac{m}{n}$ which are as given below:

$$\left(\frac{kx_2+x_1}{1+k}, \frac{ky_2+y_1}{1+k}, \frac{kz_2+z_1}{1+k}\right)$$

Generally, this result is used in solving problems involving a general point on the line passing through two given points.

Example 7 Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 (i) internally, and (ii) externally.

Solution (i) Let P (x, y, z) be the point which divides line segment joining A(1, – 2, 3) and B (3, 4, –5) internally in the ratio 2 : 3. Therefore

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5}, \quad y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5}, \quad z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$

(ii) Let P (x, y, z) be the point which divides segment joining A (1, –2, 3) and B (3, 4, –5) externally in the ratio 2 : 3. Then

$$x = \frac{2(3)+(-3)(1)}{2+(-3)} = -3, \quad y = \frac{2(4)+(-3)(-2)}{2+(-3)} = -14, \quad z = \frac{2(-5)+(-3)(3)}{2+(-3)} = 19$$

Therefore, the required point is (–3, –14, 19).

Example 8 Using section formula, prove that the three points (– 4, 6, 10), (2, 4, 6) and (14, 0, –2) are collinear.

Solution Let A (– 4, 6, 10), B (2, 4, 6) and C(14, 0, – 2) be the given points. Let the point P divides AB in the ratio $k : 1$. Then coordinates of the point P are

$$\left(\frac{2k-4}{k+1}, \frac{4k+6}{k+1}, \frac{6k+10}{k+1}\right)$$

Let us examine whether for some value of k, the point P coincides with point C.

On putting $\frac{2k-4}{k+1} = 14$, we get $k = -\frac{3}{2}$

When $k = -\frac{3}{2}$, then $\frac{4k+6}{k+1} = \frac{4(-\frac{3}{2})+6}{-\frac{3}{2}+1} = 0$

and $$\frac{6k+10}{k+1} = \frac{6(-\frac{3}{2})+10}{-\frac{3}{2}+1} = -2$$

Therefore, C (14, 0, –2) is a point which divides AB externally in the ratio 3 : 2 and is same as P.Hence A, B, C are collinear.

Example 9 Find the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Solution Let ABC be the triangle. Let the coordinates of the vertices A, B,C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), respectively. Let D be the mid-point of BC. Hence coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

Let G be the centroid of the triangle. Therefore, it divides the median AD in the ratio 2 : 1. Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}, \frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

or $$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Example 10 Find the ratio in which the line segment joining the points (4, 8, 10) and (6, 10, – 8) is divided by the YZ-plane.

Solution Let YZ-plane divides the line segment joining A (4, 8, 10) and B (6, 10, – 8) at P (x, y, z) in the ratio k : 1. Then the coordinates of P are

$$\left(\frac{4+6k}{k+1}, \frac{8+10k}{k+1}, \frac{10-8k}{k+1}\right)$$

Since P lies on the YZ-plane, its x-coordinate is zero, i.e., $\frac{4+6k}{k+1}=0$

or $k=-\frac{2}{3}$

Therefore, YZ-plane divides AB externally in the ratio 2 : 3.

EXERCISE 12.3

1. Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, – 4, 6) in the ratio (i) 2 : 3 internally, (ii) 2 : 3 externally.
2. Given that P (3, 2, – 4), Q (5, 4, – 6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.
3. Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).
4. Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and $C\left(0,\frac{1}{3},2\right)$ are collinear.
5. Find the coordinates of the points which trisect the line segment joining the points P (4, 2, – 6) and Q (10, –16, 6).

Chapter 14: Probability

Random experiments; outcomes, sample space (set representation).

16.2 Random Experiments

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called *random experiments*.

An experiment is called random experiment if it satisfies the following two conditions:

(i) It has more than one possible outcome.

(ii) It is not possible to predict the outcome in advance.

Check whether the experiment of tossing a die is random or not?

In this chapter, we shall refer the random experiment by experiment only unless stated otherwise.

16.2.1 ***Outcomes and sample space*** A possible result of a random experiment is called its *outcome*.

Consider the experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5, or 6, if we are interested in the number of dots on the upper face of the die.

The set of outcomes {1, 2, 3, 4, 5, 6} is called the *sample space of the experiment*.

Thus, the set of all possible outcomes of a random experiment is called the *sample space* associated with the experiment. Sample space is denoted by the symbol S.

Each element of the sample space is called a *sample point*. In other words, each outcome of the random experiment is also called *sample point*.

Let us now consider some examples.

Example 1 Two coins (a one rupee coin and a two rupee coin) are tossed once. Find a sample space.

Solution Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin. Since either coin can turn up Head (H) or Tail(T), the possible outcomes may be

Heads on both coins = (H,H) = HH
Head on first coin and Tail on the other = (H,T) = HT
Tail on first coin and Head on the other = (T,H) = TH
Tail on both coins = (T,T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

Note The outcomes of this experiment are ordered pairs of H and T. For the sake of simplicity the commas are omitted from the ordered pairs.

Example 2 Find the sample space associated with the experiment of rolling a pair of dice (one is blue and the other red) once. Also, find the number of elements of this sample space.

Solution Suppose 1 appears on blue die and 2 on the red die. We denote this outcome by an ordered pair (1,2). Similarly, if '3' appears on blue die and '5' on red, the outcome is denoted by the ordered pair (3,5).

In general each outcome can be denoted by the ordered pair (x, y), where x is the number appeared on the blue die and y is the number appeared on the red die. Therefore, this sample space is given by

S = {(x, y): x is the number on the blue die and y is the number on the red die}.

The number of elements of this sample space is $6 \times 6 = 36$ and the sample space is given below:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Example 3 In each of the following experiments specify appropriate sample space

(i) A boy has a 1 rupee coin, a 2 rupee coin and a 5 rupee coin in his pocket. He takes out two coins out of his pocket, one after the other.

(ii) A person is noting down the number of accidents along a busy highway during a year.

Solution (i) Let Q denote a 1 rupee coin, H denotes a 2 rupee coin and R denotes a 5 rupee coin. The first coin he takes out of his pocket may be any one of the three coins Q, H or R. Corresponding to Q, the second draw may be H or R. So the result of two draws may be QH or QR. Similarly, corresponding to H, the second draw may be Q or R.

Therefore, the outcomes may be HQ or HR. Lastly, corresponding to R, the second draw may be H or Q.

So, the outcomes may be RH or RQ.

Thus, the sample space is S={QH, QR, HQ, HR, RH, RQ}

(ii) The number of accidents along a busy highway during the year of observation can be either 0 (for no accident) or 1 or 2, or some other positive integer.

Thus, a sample space associated with this experiment is S= {0,1,2,...}

Example 4 A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Solution Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

$$S = \{ HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6\}.$$

Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, Ti means tail on the coin and the number i on the die.

Example 5 Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on till head is obtained. Hence, the desired sample space is

S= {H, TH, TTH, TTTH, TTTTH,...}

EXERCISE 16.1

In each of the following Exercises 1 to 7, describe the sample space for the indicated experiment.

1. A coin is tossed three times.
2. A die is thrown two times.
3. A coin is tossed four times.
4. A coin is tossed and a die is thrown.
5. A coin is tossed and then a die is rolled only in case a head is shown on the coin.
6. 2 boys and 2 girls are in Room X, and 1 boy and 3 girls in Room Y. Specify the sample space for the experiment in which a room is selected and then a person.
7. One die of red colour, one of white colour and one of blue colour are placed in a bag. One die is selected at random and rolled, its colour and the number on its uppermost face is noted. Describe the sample space.
8. An experiment consists of recording boy–girl composition of families with 2 children.
   (i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?

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(ii) What is the sample space if we are interested in the number of girls in the family?

9. A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.

10. An experiment consists of tossing a coin and then throwing it second time if a head occurs. If a tail occurs on the first toss, then a die is rolled once. Find the sample space.

11. Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non – defective(N). Write the sample space of this experiment.

12. A coin is tossed. If the out come is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for the experiment?

13. The numbers 1, 2, 3 and 4 are written separatly on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.

14. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.

15. A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.

16. A die is thrown repeatedly untill a six comes up. What is the sample space for this experiment?

Extra Reading Material for Chapter 2: Relations and Functions

Composition of Functions

1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions f_1, f_2, f_3 and f_4 given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of X_1 under the function f_1 are distinct, but the image of two distinct elements 1 and 2 of X_1 under f_2 is same, namely b. Further, there are some elements like e and f in X_2 which are not images of any element of X_1 under f_1, while all elements of X_3 are images of some elements of X_1 under f_3. The above observations lead to the following definitions:

Definition 5 A function $f : X \to Y$ is defined to be *one-one* (or *injective*), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called *many-one*.

The function f_1 and f_4 in Fig 1.2 (i) and (iv) are one-one and the function f_2 and f_3 in Fig 1.2 (ii) and (iii) are many-one.

Definition 6 A function $f : X \to Y$ is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Fig 1.2 (iii), (iv) are onto and the function f_1 in Fig 1.2 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

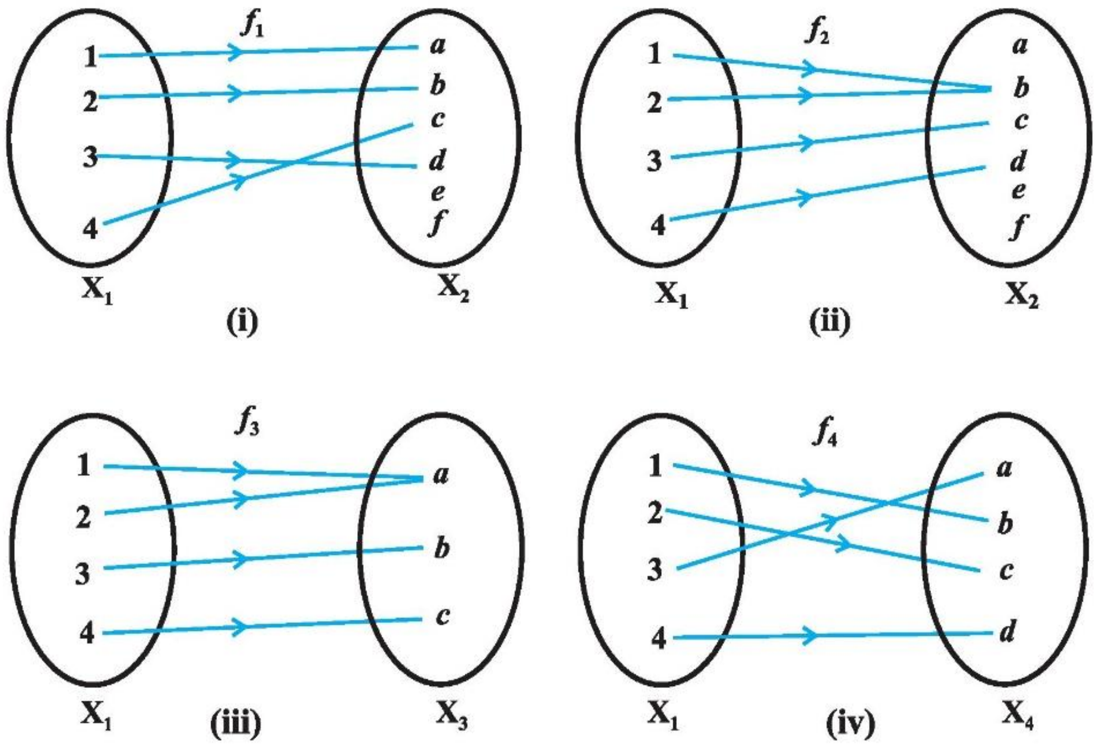


Fig 1.2 (i) to (iv)

Remark $f : X \rightarrow Y$ is onto if and only if Range of $f = Y$.

Definition 7 A function $f : X \rightarrow Y$ is said to be *one-one* and *onto* (or *bijective*), if f is both one-one and onto.

The function f_4 in Fig 1.2 (iv) is one-one and onto.

Example 7 Let A be the set of all 50 students of Class X in a school. Let $f : A \rightarrow \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

Solution No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in **N** is not roll number of any student of the class, so that 51 can not be image of any element of X under f. Hence, f is not onto.

Example 8 Show that the function $f : \mathbf{N} \rightarrow \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Solution The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Further, f is not onto, as for $1 \in \mathbf{N}$, there does not exist any x in **N** such that $f(x) = 2x = 1$.

Example 9 Prove that the function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = 2x$, is one-one and onto.

Solution f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Also, given any real number y in R, there exists $\frac{y}{2}$ in R such that $f(\frac{y}{2}) = 2 \cdot (\frac{y}{2}) = y$. Hence, f is onto.

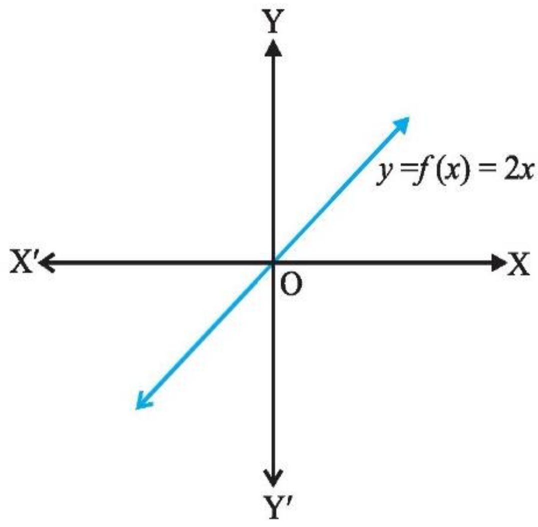


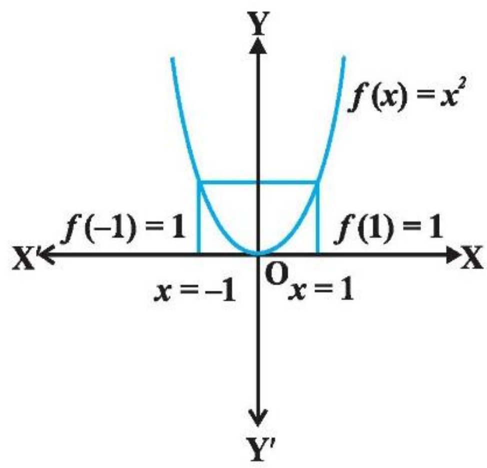
Fig 1.3

Example 10 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not one-one.

Solution f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in \mathbf{N}$, we have $f(1) = 1$.

Example 11 Show that the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x^2$, is neither one-one nor onto.

Solution Since $f(-1) = 1 = f(1)$, f is not one-one. Also, the element -2 in the co-domain $\mathbf{R}$ is not image of any element x in the domain $\mathbf{R}$ (Why?). Therefore f is not onto.



The image of 1 and –1 under f is 1.

Fig 1.4

Example 12 Show that $f : \mathbf{N} \to \mathbf{N}$, given by

$$f(x) = \begin{cases} x+1, \text{if } x \text{ is odd,} \\ x-1, \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.

Solution Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain **N** is the image of $2r + 2$ in the domain **N** and any even number $2r$ in the co-domain **N** is the image of $2r - 1$ in the domain **N**. Thus, f is onto.

Example 13 Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Solution Suppose f is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element. Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one-one.

Example 14 Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Solution Since f is one-one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

Remark The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set X, i.e., a one-one function $f : X \to X$ is necessarily onto and an onto map $f : X \to X$ is necessarily one-one, for every finite set X. In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

EXERCISE 1.2

1. Show that the function $f : \mathbf{R}_* \to \mathbf{R}_*$ defined by $f(x) = \dfrac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?
2. Check the injectivity and surjectivity of the following functions:
   (i) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
   (ii) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
   (iii) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
   (iv) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
   (v) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$
3. Prove that the Greatest Integer Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

5. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ \ 1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

   (i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

   (ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

8. Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

9. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

   State whether the function f is bijective. Justify your answer.

10. Let A = $\mathbf{R}$ – {3} and B = $\mathbf{R}$ – {1}. Consider the function $f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

    (A) f is one-one onto
    (B) f is many-one onto
    (C) f is one-one but not onto
    (D) f is neither one-one nor onto.

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

    (A) f is one-one onto
    (B) f is many-one onto
    (C) f is one-one but not onto
    (D) f is neither one-one nor onto.

1.4 Composition of Functions and Invertible Function

In this section, we will study composition of functions and the inverse of a bijective function. Consider the set A of all students, who appeared in Class X of a Board Examination in 2006. Each student appearing in the Board Examination is assigned a roll number by the Board which is written by the students in the answer script at the time of examination. In order to have confidentiality, the Board arranges to deface the roll numbers of students in the answer scripts and assigns a fake code number to each roll number. Let $B \subset \mathbf{N}$ be the set of all roll numbers and $C \subset \mathbf{N}$ be the set of all code numbers. This gives rise to two functions $f : A \to B$ and $g : B \to C$ given by $f(a)$ = the roll number assigned to the student a and $g(b)$ = the code number assigned to the roll number b. In this process each student is assigned a roll number through the function f and each roll number is assigned a code number through the function g. Thus, by the combination of these two functions, each student is eventually attached a code number.

This leads to the following definition:

Definition 8 Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)),\ \forall\ x \in A.$$

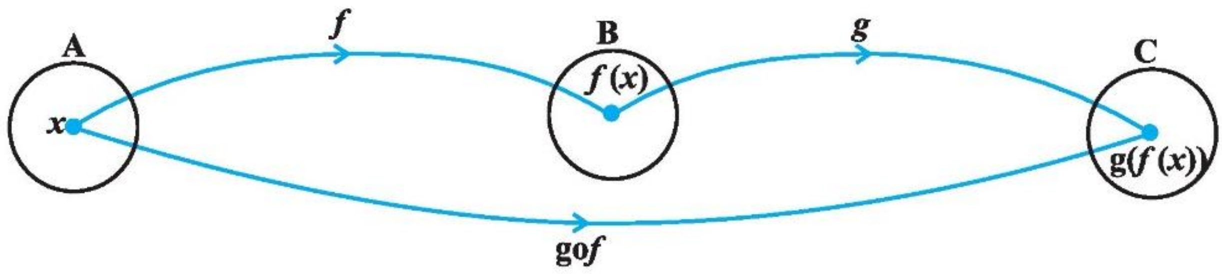


Fig 1.5

Example 15 Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Example 17 Show that if $f : \mathbf{R} - \left\{\frac{7}{5}\right\} \to \mathbf{R} - \left\{\frac{3}{5}\right\}$ is defined by $f(x) = \frac{3x+4}{5x-7}$ and $g : \mathbf{R} - \left\{\frac{3}{5}\right\} \to \mathbf{R} - \left\{\frac{7}{5}\right\}$ is defined by $g(x) = \frac{7x+4}{5x-3}$, then $fog = \mathrm{I}_A$ and $gof = \mathrm{I}_B$, where, $A = \mathbf{R} - \left\{\frac{3}{5}\right\}$, $B = \mathbf{R} - \left\{\frac{7}{5}\right\}$; $\mathrm{I}_A(x) = x$, $\forall\, x \in A$, $\mathrm{I}_B(x) = x$, $\forall\, x \in B$ are called identity functions on sets A and B, respectively.

Solution We have

$$gof(x) = g\left(\frac{3x+4}{5x-7}\right) = \frac{7\left(\frac{(3x+4)}{(5x-7)}\right)+4}{5\left(\frac{(3x+4)}{(5x-7)}\right)-3} = \frac{21x+28+20x-28}{15x+20-15x+21} = \frac{41x}{41} = x$$

Similarly, $$fog(x) = f\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{(7x+4)}{(5x-3)}\right)+4}{5\left(\frac{(7x+4)}{(5x-3)}\right)-7} = \frac{21x+12+20x-12}{35x+20-35x+21} = \frac{41x}{41} = x$$

Thus, $gof(x) = x$, $\forall\, x \in B$ and $fog(x) = x$, $\forall\, x \in A$, which implies that $gof = \mathrm{I}_B$ and $fog = \mathrm{I}_A$.

Example 18 Show that if $f : A \to B$ and $g : B \to C$ are one-one, then $gof : A \to C$ is also one-one.

Solution Suppose $gof(x_1) = gof(x_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad f(x_1) = f(x_2)$, as g is one-one

$\Rightarrow \quad x_1 = x_2$, as f is one-one

Hence, gof is one-one.

Example 19 Show that if $f : A \to B$ and $g : B \to C$ are onto, then $gof : A \to C$ is also onto.

Solution Given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto. Further, for $y \in B$, there exists an element x in A with $f(x) = y$, since f is onto. Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

Example 20 Consider functions f and g such that composite gof is defined and is one-one. Are f and g both necessarily one-one.

Solution Consider $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$ defined as $f(x) = x$, $\forall x$ and $g : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ as $g(x) = x$, for $x = 1, 2, 3, 4$ and $g(5) = g(6) = 5$. Then, $gof(x) = x \ \forall x$, which shows that gof is one-one. But g is clearly not one-one.

Example 21 Are f and g both necessarily onto, if gof is onto?

Solution Consider $f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ and $g : \{1, 2, 3, 4\} \to \{1, 2, 3\}$ defined as $f(1) = 1, f(2) = 2$, $f(3) = f(4) = 3$, $g(1) = 1$, $g(2) = 2$ and $g(3) = g(4) = 3$. It can be seen that gof is onto but f is not onto.

Remark It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

Now, we would like to have close look at the functions f and g described in the beginning of this section in reference to a Board Examination. Each student appearing in Class X Examination of the Board is assigned a roll number under the function f and each roll number is assigned a code number under g. After the answer scripts are examined, examiner enters the mark against each code number in a mark book and submits to the office of the Board. The Board officials decode by assigning roll number back to each code number through a process reverse to g and thus mark gets attached to roll number rather than code number. Further, the process reverse to f assigns a roll number to the student having that roll number. This helps in assigning mark to the student scoring that mark. We observe that while composing f and g, to get gof, first f and then g was applied, while in the reverse process of the composite gof, first the reverse process of g is applied and then the reverse process of f.

Example 22 Let $f : \{1, 2, 3\} \to \{a, b, c\}$ be one-one and onto function given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that there exists a function $g : \{a, b, c\} \to \{1, 2, 3\}$ such that $gof = I_X$ and $fog = I_Y$, where, $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Solution Consider $g : \{a, b, c\} \to \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$ and $g(c) = 3$. It is easy to verify that the composite $gof = I_X$ is the identity function on X and the composite $fog = I_Y$ is the identity function on Y.

Remark The interesting fact is that the result mentioned in the above example is true for an arbitrary one-one and onto function $f : X \to Y$. Not only this, even the converse is also true , i.e., if $f : X \to Y$ is a function such that there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$, then f must be one-one and onto.

The above discussion, Example 22 and Remark lead to the following definition:

Definition 9 A function $f : X \rightarrow Y$ is defined to be *invertible*, if there exists a function $g : Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the *inverse of f* and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 23 Let $f : \mathbf{N} \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where, $Y = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$, for some x in the domain $\mathbf{N}$. This shows that $x = \frac{(y-3)}{4}$. Define $g : Y \rightarrow \mathbf{N}$ by $g(y) = \frac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$ and

$fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y$. This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible and g is the inverse of f.

Example 24 Let $Y = \{n^2 : n \in \mathbf{N}\} \subset \mathbf{N}$. Consider $f : \mathbf{N} \rightarrow Y$ as $f(n) = n^2$. Show that f is invertible. Find the inverse of f.

Solution An arbitrary element y in Y is of the form n^2, for some $n \in \mathbf{N}$. This implies that $n = \sqrt{y}$. This gives a function $g : Y \rightarrow \mathbf{N}$, defined by $g(y) = \sqrt{y}$. Now, $gof(n) = g(n^2) = \sqrt{n^2} = n$ and $fog(y) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$, which shows that $gof = I_N$ and $fog = I_Y$. Hence, f is invertible with $f^{-1} = g$.

Example 25 Let $f' : \mathbf{N} \rightarrow \mathbf{R}$ be a function defined as $f'(x) = 4x^2 + 12x + 15$. Show that $f : \mathbf{N} \rightarrow S$, where, S is the range of f, is invertible. Find the inverse of f.

Solution Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in $\mathbf{N}$, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}$, as $y \geq 6$.

Let us define $g : \text{S} \to \mathbf{N}$ by $g(y) = \frac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}$.

Now $\qquad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$$= \frac{\left(\left(\sqrt{(2x+3)^2+6-6}\right)-3\right)}{2} = \frac{(2x+3-3)}{2} = x$$

and $\qquad fog\ (y) = f\left(\frac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}\right) = \left(\frac{2\left(\left(\sqrt{y-6}\right)-3\right)}{2}+3\right)^2 + 6$

$$= \left(\left(\sqrt{y-6}\right)-3+3\right)^2 + 6 = \left(\sqrt{y-6}\right)^2 + 6 = y - 6 + 6 = y.$$

Hence, $\qquad gof = \text{I}_{\mathbf{N}}$ and $fog = \text{I}_{\text{S}}$. This implies that f is invertible with $f^{-1} = g$.

Example 26 Consider $f : \mathbf{N} \to \mathbf{N}$, $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{R}$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall\, x, y$ and z in N. Show that $ho(gof) = (hog)\, of$.

Solution We have

$$ho(gof)\ (x) = h(gof\ (x)) = h(g(f(x))) = h(g(2x))$$
$$= h(3(2x) + 4) = h(6x + 4) = \sin\ (6x + 4)\ \ \forall\, x \in \mathbf{N}.$$

Also, $\qquad ((hog)\,o f\,)\ (x) = (hog)\ (f(x)) = (hog)\ (2x) = h(g(2x))$

$$= h(3(2x) + 4) = h(6x + 4) = \sin\ (6x + 4),\ \forall\, x \in \mathbf{N}.$$

This shows that $ho(gof) = (hog)\,o f$.

This result is true in general situation as well.

Theorem 1 If $f : \text{X} \to \text{Y}$, $g : \text{Y} \to \text{Z}$ and $h : \text{Z} \to \text{S}$ are functions, then

$$ho(gof) = (hog)\,o f.$$

Proof We have

$$ho(gof)\ (x) = h(gof(x)) = h(g(f(x))),\ \forall\, x \text{ in X}$$

and $\qquad (hog)\ of\ (x) = hog\,(f\ (x)) = h(g(f(x))),\ \forall\, x$ in X.

Hence, $\qquad ho(gof) = (hog)\,o f.$

Example 27 Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to \{\text{apple, ball, cat}\}$ defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a) =$ apple, $g(b) =$ ball and $g(c) =$ cat. Show that f, g and gof are invertible. Find out f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1} o\, g^{-1}$.

Solution Note that by definition, f and g are bijective functions. Let f^{-1}: $\{a, b, c\} \to (1, 2, 3\}$ and g^{-1} : {apple, ball, cat} $\to \{a, b, c\}$ be defined as $f^{-1}\{a\} = 1, f^{-1}\{b\} = 2,\ f^{-1}\{c\} = 3,\ g^{-1}$\{apple\} $= a,\ g^{-1}$\{ball\} $= b$ and g^{-1}\{cat\} $= c$. It is easy to verify that $f^{-1}of = I_{\{1, 2, 3\}}, f\,o\,f^{-1} = I_{\{a, b, c\}}, g^{-1}og = I_{\{a, b, c\}}$ and $g\,o\,g^{-1} = I_D$, where, D = {apple, ball, cat}. Now, gof : {1, 2, 3} → {apple, ball, cat} is given by $gof(1)$ = apple, $gof(2)$ = ball, $gof(3)$ = cat. We can define

$(gof)^{-1}$: {apple, ball, cat} → {1, 2, 3} by $(gof)^{-1}$ (apple) = 1, $(gof)^{-1}$ (ball) = 2 and $(g\,of)^{-1}$ (cat) = 3. It is easy to see that $(g\,o\,f)^{-1}\ o\ (g\,o\,f) = I_{\{1, 2, 3\}}$ and $(gof)\ o\ (gof)^{-1} = I_D$. Thus, we have seen that f, g and gof are invertible.

Now, $f^{-1}og^{-1}$ (apple)= $f^{-1}(g^{-1}$(apple)) $= f^{-1}(a) = 1 = (gof)^{-1}$ (apple)

$f^{-1}og^{-1}$ (ball) $= f^{-1}(g^{-1}$(ball)) $= f^{-1}(b) = 2 = (gof)^{-1}$ (ball) and

$f^{-1}og^{-1}$ (cat) $= f^{-1}(g^{-1}$(cat)) $= f^{-1}(c) = 3 = (gof)^{-1}$ (cat).

Hence $(gof)^{-1} = f^{-1}og^{-1}$.

The above result is true in general situation also.

Theorem 2 Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

Proof To show that gof is invertible with $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that $(f^{-1}og^{-1})o(gof) = I_X$ and $(gof)o(f^{-1}og^{-1}) = I_Z$.

Now, $(f^{-1}og^{-1})o(gof) = ((f^{-1}og^{-1})\ og)\ of$, by Theorem 1

$= (f^{-1}o(g^{-1}og))\ of$, by Theorem 1

$= (f^{-1}\,o\,I_Y)\ of$, by definition of g^{-1}

$= I_X$.

Similarly, it can be shown that $(gof\,)\,o\,(f^{-1}\,o\,g^{-1}) = I_Z$.

Example 28 Let S = {1, 2, 3}. Determine whether the functions $f : S \to S$ defined as below have inverses. Find f^{-1}, if it exists.

(a) $f = \{(1, 1), (2, 2), (3, 3)\}$
(b) $f = \{(1, 2), (2, 1), (3, 1)\}$
(c) $f = \{(1, 3), (3, 2), (2, 1)\}$

Solution

(a) It is easy to see that f is one-one and onto, so that f is invertible with the inverse f^{-1} of f given by $f^{-1} = \{(1, 1), (2, 2), (3, 3)\} = f$.
(b) Since $f(2) = f(3) = 1$, f is not one-one, so that f is not invertible.
(c) It is easy to see that f is one-one and onto, so that f is invertible with $f^{-1} = \{(3, 1), (2, 3), (1, 2)\}$.

EXERCISE 1.3

1. Let $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

2. Let f, g and h be functions from **R** to **R**. Show that

$$(f + g)oh = foh + goh$$
$$(f \,.\, g)oh = (foh) \,.\, (goh)$$

3. Find gof and fog, if

(i) $f(x) = |\,x\,|$ and $g(x) = |\,5x - 2\,|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

4. If $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

5. State with reason whether following functions have inverse

(i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ with
$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with
$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with
$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

6. Show that $f: [-1, 1] \rightarrow \mathbf{R}$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f: [-1, 1] \rightarrow$ Range f.

(Hint: For $y \in$ Range f, $y = f(x) = \dfrac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \dfrac{2y}{(1-y)}$)

7. Consider $f: \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

8. Consider $f: \mathbf{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

9. Consider $f : \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

10. Let $f : X \to Y$ be an invertible function. Show that f has unique inverse. (Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $fog_1(y) = 1_Y(y) = fog_2(y)$. Use one-one ness of f).

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

12. Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

13. If $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3-x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $x^{\frac{1}{3}}$ (B) x^3 (C) x (D) $(3 - x^3)$.

14. Let $f : \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map g : Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$

Extra Reading Material for Chapter 12: Limits and Derivatives

Derivatives of composite functions (Chain rule).

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3 + 12x^2 + 6x + 1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\ (2x + 1)^2$$

Now, observe that $\quad f(x) = (h \text{ o } g)\ (x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\ (2x + 1)^2 = 3(2x + 1)^2\ .\ 2 = 3t^2\ .\ 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e., $f = (w \text{ o } u) \text{ o } v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \text{o} u)}{dt}\cdot\frac{dt}{dx} = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \text{ o } u)\ (x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$y = \sin (x^2) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2)$$

$$= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2$$

Example 22 Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan (2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan (2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2 (2x+3)$$

Example 23 Differentiate $\sin (\cos (x^2))$ with respect to x.

Solution The function $f(x) = \sin (\cos (x^2))$ is a composition $f(x) = (w \circ v \circ u)(x)$ of the three functions u, v and w, where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put $t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s, \frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$ exist for all real x. Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) \,.\, (-\sin t) \,.\, (2x) = -2x \sin x^2 \,.\, \cos (\cos x^2)$$

Alternatively, we can proceed as follows:

$$y = \sin (\cos x^2)$$

Therefore $$\frac{dy}{dx} = \frac{d}{dx} \sin (\cos x^2) = \cos (\cos x^2) \frac{d}{dx} (\cos x^2)$$

$$= \cos (\cos x^2) (- \sin x^2) \frac{d}{dx} (x^2)$$

$$= - \sin x^2 \cos (\cos x^2) (2x)$$

$$= - 2x \sin x^2 \cos (\cos x^2)$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin (x^2 + 5)$ **2.** $\cos (\sin x)$ **3.** $\sin (ax + b)$

4. $\sec (\tan (\sqrt{x}))$ **5.** $\frac{\sin (ax+b)}{\cos (cx+d)}$ **6.** $\cos x^3 . \sin^2 (x^5)$

7. $2\sqrt{\cot(x^2)}$ **8.** $\cos(\sqrt{x})$

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