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PBMT - 02

Sec B

Q11

$$x^y = y^x$$

take log

$$y \log x = x \log y$$

dwr to x

$$\frac{y}{x} + \frac{dy}{dx}\log x = \log y + \frac{x}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\log x - \frac{x}{y}\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\log y - y/x}{\log x - x/y}$$

Q12

$r = 10\ cm$

$$\text{volume} = \frac{4}{3}\pi r^3$$

dwr to ~~radius~~ radius

$$\frac{dv}{dr} = \frac{4}{3} \times 3\pi r^2$$

$$= 4\pi(10)(10)$$

$$= 400\pi\ cm^3/cm$$

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Q13 (a) $\tan^{-1}(x^3 + y^3) = a^{2025}$

dwr to x

$$\frac{1}{1 + x^3 + y^3} \times 3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} = 1 + x^3 + y^3$$

$$3y^2\frac{dy}{dx} = 1 + x^3 + y^3 - 3x^2$$

$$\frac{dy}{dx} = \frac{1 + x^3 + y^3 - 3x^2}{3y^2}$$

or.

(b) $y = 5\cos x - 3\sin x$

dwr to x

$$y' = -5\sin x - 3\cos x.$$

dwr to x.

$$y'' = -5\cos x + 3\sin x$$

$$y'' = -[5\cos x - 3\sin x]$$

$$y'' = -y$$

$$y'' + y = 0.$$

Sec C

Q14 $y = \dfrac{\sin x \cos x}{\sin x + \cos x}$

dwr to x

$$y' = \frac{(\sin x + \cos x)(-\sin^2 x + \cos^2 x) - (\sin x \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x)(\cos^2 x - \sin^2 x) - (\sin x \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\cancel{\sin}(\cos x - \sin x)\left[(\sin x + \cos x)^2 - \sin x \cos x\right]}{(\sin x + \cos x)^2}$$

14. $f(x) = \dfrac{\sin x \cos x}{\sin x + \cos x}$

$$f'(x) = \frac{(\sin x+\cos x)(-\sin^2 x+\cos^2 x) - \sin x\cos x(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$= \frac{(\sin x+\cos x)(\cos^2 x-\sin^2 x) - \sin x\cos x(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$= \frac{(\cos x-\sin x)[(\sin x+\cos x)^2 - \sin x\cos x]}{(\sin x+\cos x)^2}$$

$$= \frac{(\cos x-\sin x)(\sin^2 x+\cos^2 x+2\sin x\cos x-\sin x\cos x)}{(\sin x+\cos x)^2}$$

$$= \frac{(\cos x-\sin x)(1+\sin x\cos x)}{(\sin x+\cos x)^2}$$

Put $f'(x) = 0$

$\cos x - \sin x = 0$ or $1 + \sin x\cos x = 0$

$\sin x = \cos x$ | $\sin x \cos x = -1$

$\tan x = 1$ | $2\sin x\cos x = -2$

$x = \pi/4$ | $\sin 2x = -2$

not possible as range of sin is $[-1,1]$

first derivative test

At $x = \pi/4$

for $x < \pi/4$

$f'(x) > 0$ ($\because \cos x > \sin x$, $\cos x - \sin x > 0$)

for $x > \pi/4$

$f'(x) < 0$ ($\because \cos x < \sin x$, $\cos x - \sin x < 0$)

As f inc then dec, indicates a local maximum at $x = \pi/4$

Max. value =

$$f(\pi/4) = \frac{\sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}\right)}{\cos\frac{\pi}{4} + \sin\left(\frac{\pi}{4}\right)} = \frac{\frac{1}{2}}{\frac{2}{\sqrt{2}}} = \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$$

Q15 $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

$x = \sin\theta \qquad y = \sin\phi$

$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = a\left[\cancel{\sin}\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]$

$$\frac{\cancel{\cos\left(\frac{\theta+\phi}{2}\right)}\cos\left(\frac{\theta-\phi}{2}\right)}{\cancel{\cos\left(\frac{\theta+\phi}{2}\right)}\sin\left(\frac{\theta-\phi}{2}\right)} = a$$

$\cot\left(\frac{\theta-\phi}{2}\right) = a.$

~~dwr to x~~

$\frac{\theta-\phi}{2} = \cot^{-1}a$

$\frac{\sin^{-1}x}{2} - \frac{\cancel{\cos^{-1}}\sin^{-1}y}{2} = \cot^{-1}a$

dwr to x.

$\frac{1}{2}\left[\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}\right] = 0$

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Hence proved.

Sec D.

Q16 $(x-a)^2 + (y-b)^2 = c^2$

dwr to x

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\cancel{2}(y-b)\frac{dy}{dx} = -\cancel{2}(x-a)$$

$$\frac{dy}{dx} = -\frac{(x-a)}{y-b}$$

dwr to x.

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b) - (x-a)\frac{dy}{dx}}{(y-b)^2}\right].$$

$$= -\frac{\left[(y-b) - (x-a)\left[-\frac{(x-a)}{y-b}\right]\right]}{(y-b)^2}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]}{(y-b)^3}$$

$$= -\frac{[c^2]}{(y-b)^3}$$

Consider:

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

$$\frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}.$$

$$\Rightarrow \frac{\left((y-b)^2 + (x-a)^2\right)^{3/2}}{\left((y-b)^2\right)^{3/2}} \times \frac{(y-b)^3}{(-c^2)}.$$

$$\Rightarrow \frac{(c^2)^{3/2}}{\cancel{(y-b)^3}} \times \frac{\cancel{(y-b)^3}}{(-c^2)}$$

$$\Rightarrow \frac{c^3}{-c^2} = -c.$$

~~Q17 (a) $F(x) = 4\sin x +$~~

17.

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$$

$$= \frac{4\sin x}{2+\cos x} - x$$

$$\therefore f(x) = \frac{(2+\cos x)\,4\cos x - 4\sin x\,(-\sin x)}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{2+\cos x}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$= \frac{\cos x\,(4-\cos x)}{(2+\cos x)^2}$$

because $-1 \le \cos x \le 1$

$\Rightarrow 4 - \cos x > 0$ and $(2 + \cos x)^2 > 0$

$\therefore$ f(x) > 0 or < 0 such that cos x > 0 or cos x < 0 respectively

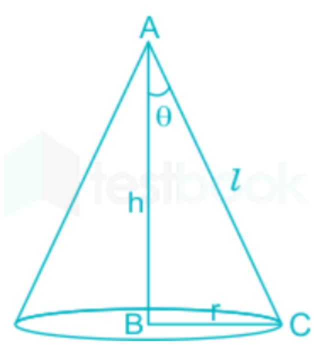
$\therefore$ f(x) is increasing when $0 < x < \frac{\pi}{2}, \frac{3\pi}{2} < x < 2\pi$

And f(x) is decreasing when $\frac{\pi}{2} < \pi < \frac{3\pi}{2}$.

or

The volume of the right circular cone is given by -

$V = \frac{1}{3}\,\pi r^2 h$



In ΔABC

r = l × sin θ and h = l × cos θ

$\therefore V = \frac{1}{3}\,\pi\,(l\sin\theta)^2\,(l\cos\theta) \Rightarrow \frac{1}{3}\,\pi l^3 \sin^2\theta\cos\theta$

$\frac{dV}{d\theta} = 0$

$\left(\frac{1}{3}\ \pi l^3 \sin^2\theta\right)(-\sin\theta) + \left(\frac{1}{3}\ \pi l^3 \cos\theta\right)(2\sin\theta\cos\theta) = 0$

$\left(\frac{1}{3}\ \pi l^3\right)\left(-sin^3\theta + 2\sin\theta\cos^2\theta\right) = 0$

$-\sin\theta\left(\sin^2\theta - 2\cos^2\theta\right) = 0$

If sin θ = 0

⇒ θ = 0° which is not possible

∴ $\sin^2\theta - 2\cos^2\theta = 0$

$\frac{\sin^2\theta}{\cos^2\theta} = 2 \ldots\ldots (i)$

$\tan^2\theta = 2$

$\tan\theta = \sqrt{2}$

$\frac{d^2V}{d\theta^2}$

$\frac{dV}{d\theta} = \left(\frac{1}{3}\ \pi l^3\right)\left(-sin^3\theta + 2\sin\theta\cos^2\theta\right)$

$\frac{1}{3}\ \pi l^3\left(-3\sin^2\theta\cos\theta + 2\sin\theta\cdot 2\cos\theta(-\sin\theta) + 2\cos^2\theta\cdot\cos\theta\right)$

$\left(-3\sin^2\theta\cos\theta + 2\sin\theta\cdot 2\cos\theta(-\sin\theta) + 2\cos^2\theta\cdot\cos\theta\right)$

$-3\sin^2\theta\cos\theta - 4\sin^2\theta\cos\theta + 2\cos^3\theta$

$Using\ \frac{\sin^2\theta}{\cos^2\theta} = 2$

$((-3)\times(2\cos^2\theta\cos\theta)) - ((4)\times(2\cos^2\theta\cos\theta)) + 2\cos^3\theta$

$-6\cos^3\theta - 8\cos^3\theta + 2\cos^3\theta$

$-12\cos^3\theta$

Semi-vertical angle θ is less than 90° i.e it lies in 1st quadrant.

cos θ is positive in 1st quadrant

$\therefore \frac{d^2V}{d\theta^2} < 0$

∴ The volume will be maximum when $\theta = \tan^{-1}\sqrt{2}$

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Sec B

QTS =

Sec A

Q1 = $\lim_{h\to0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to0} \sqrt{\ }$

$\lim_{h\to0} f(-h) = \lim_{h\to0} f(h) = f(0)$

$\lim_{h\to0} \frac{\sqrt{4-h}-2}{-h} = \lim_{h\to0} \frac{\sqrt{4+h}-2}{h} = k$

$\lim_{h\to0} \frac{\sqrt{4+h}-2}{h} = k$

$\lim_{h\to0} \frac{\sqrt{4+h}-2}{h} \times \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$

$\lim_{h\to0} \frac{(4+h)-4}{h\sqrt{4+h}+2)} = k$

$\lim_{h\to0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} = k$

$\lim_{h\to0} \frac{1}{\sqrt{4+h}+2} = k$

$\frac{1}{\sqrt{4}+2} = k$

$\frac{1}{2+2} = k$

$\frac{1}{4} = k$

(b)

Q2 = $f'(x) = \cancel{6x^2}\ 6x^2+18x+12$

$= 3(2x^2+9x+6)$

$f'(x) = 0$

$0 = 2x^2+9x+6$

$2(0)+9(0)+6 = 6$

$2(-1.5)^2+9(-1.5)+6$

$2(2.25) - 13.5+6$

$4.50+6-13.5$

$10.50-13.5$

$-3.$

$2(-3)^2+9(-3)+6$

$18 - 27+6$

$24-27 = -3.$

$\frac{7\times(11)}{\frac{48}{3}}$ $81-4(7)(6)$
$81-48$

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$0 = 2x^2+9x+6$
$= 2x^2 +$

Q2 $f'(x) = 6x^2 - 18x + 12$
$= 6(x^2-3x+2)$
$= 6(x-1)(x-2)$
$x = 1, 2$
(a) $(-1, \infty)$

Q3 = $f'(x) = 2x-6$
$0 = 2x-6$
$0 = 2(x-3)$
$x = 3$

$(-\infty, 3)$ $2(2)-6 = -2$ dec.
$(3, \infty)$ $= 2(4)-6 = 2$ inc.

(b) $(3, \infty)$

Q4 $f(x) = 5 + |\sin x - 2|$
$-1 \le \sin x \le 1$
$-1-2 \le \sin x - 2 \le 1-2$
$-3 \le \sin x - 2 \le -1$
$3 \le |\sin x - 2| \le 1$
$8 \le 5 + |\sin x - 2| \le 9$
(d) 9.

Q5 = $\lim_{x\to-\frac{\pi}{3}} \frac{\sqrt{3}\cos x + \sin x}{x+\frac{\pi}{3}} = k$

$2 \lim_{x\to-\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x}{x+\pi/3} = k$

$2 \lim_{x\to0} \frac{\sin\left[x+\frac{\pi}{3}\right]}{x+\frac{\pi}{3}} = k$

$2(1) = k$
(c) 2.

$2\sin^2 x = 1-\cos 2x \quad \cos 2x = 2\cos^2 x - 1$
$= 2\sin 1 - 2\sin^2 x$

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Q6 = $y = \tan^{-1}\left(\frac{1-\cos 2x}{\sin 2x}\right)$

$\tan y = 2\sin^2 x$

$y = \tan^{-1}\left(\frac{2\sin^2 x}{2\cos x \sin x}\right)$

$= \tan^{-1}(\tan x)$

$y = x$

dwr

$\frac{dy}{dx} = 1$ (a) 1

Q7 = $x = 1.5$

Q8 = $x+y=3$ $\quad xy^2$ - Max.

$x = 3-y$ $\quad (3-y)y^2$

$f(y) = 3y^2 - y^3$

dwr to y

$f'(y) = 6y - 3y^2$

$f'(y) = 0$

$0 = 6y - 3y^2$

$0 = 2y - y^2$

$0 = y(2-y)$

$y = 0, 2.$

$f''(y) = 6 - 6y$

$f''(0) = 0$

$f''(2) = 6 - 12 = -6 < 0$

max at $y = 2$.

$x = 3-2 = 1$

Product max value $= xy^2 = (1)(2)^2 = 4$

(b)

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Q9 = $f(x) = 2x^3 - 9x^2 + 12x + 4$

$f'(x) = 6x^2 - 18x + 12$

$0 = 6(x^2 - 3x + 2)$

$= 6(x^2 - 2x - x + 2)$

$= 6(x-2)(x-1)$

$x = 2, 1, 0$

~~[0, 1]~~

(d) $1 < x < 2$.

Q10 $\frac{ds}{dt} = 0.2$

$P = 4S$

$\frac{dP}{dt} = 4\frac{ds}{dt} = 4(0.2) = 0.8.$

(a).

Sec E

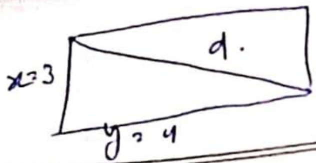
Q18 (i) $\frac{dx}{dt} = -6$ cm/s $\quad \frac{dy}{dt} = 5$ cm/s

(ii) $P = 2(x+y)$

$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-6+5)$

$= 2(-1)$

$= -2$ cm/s.



(i

(ii) $x^2 + y^2 = d^2$

$\sqrt{x^2+y^2} = d$

dwr to t

$\sqrt{2x\frac{dx}{dt} + 2y\frac{dy}{dt}} = \frac{\cancel{dt}\,\cancel{dd}}{d}\; f'(d).$

$\sqrt{2(3)(-6) + 2(4)(5)}$

$\sqrt{-36+40}$

$\sqrt{4} = \underline{2\ \text{cm/s}}.$

or.

The critical point of the function is $\mathbf{x = 5}$.

$\Rightarrow \dfrac{y\frac{dx}{dt} - x\frac{dy}{dt}}{y^2}$

$\Rightarrow \dfrac{4(-6) - 3(5)}{4\times 4} = \dfrac{-24-15}{16}$

$= \underline{\dfrac{-39}{16}}.$

19 (i) The area of the gate as a function of x is $\mathbf{A(x) = (10 + x)\sqrt{100 - x^2}}$.

(ii) The critical point of the function is $\mathbf{x = 5}$.

(iii) The maximum area of the gate is $\mathbf{75\sqrt{3}}\ \mathrm{m}^2$.