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6. Application of Derivative

(Class XII CBSE Board Exam Models from 2022-2025 with solutions)

6.1 Rate of change :

a. Algebraic :

1.

Find the points on the curve $6y = x^3 + 2$ at which ordinate is changing 8 times as fast as abscissa.

Sol.

Differentiating both sides w.r.to t, we get $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

$6 \cdot 8\frac{dx}{dt} = 3x^2\frac{dx}{dt} \Rightarrow x = \pm 4$

Points are $(4, 11)$ and $(-4, -\frac{31}{3})$

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2.

Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

Sol.

Here, $\frac{dx}{dt} = \frac{dy}{dt}$

Given $y^2 = 8x$ gives $2y.\frac{dy}{dt} = 8\frac{dx}{dt}$

$\Rightarrow 2y = 8$ or $y = 4$

Also, $y = 4$ gives $x = 2$.

Thus, the point $(2, 4)$

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3.

A particle moves along the curve $3y = ax^3 + 1$ such that at a point with x–coordinate 1, y–coordinate is changing twice as fast at x–coordinate. Find the value of a.

Sol.

Differentiating equation $3y = ax^3 + 1$ with respect to 'x', $3\frac{dy}{dx} = 3ax^2$

Taking $x = 1$, $\frac{dy}{dx} = 2$, $3(2) = 3a(1)^2 \Rightarrow a = 2$

4.

For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/s, then how fast is the slope of the curve changing when $x = 2$?

Sol.

$y = 5x - 2x^3$

Given $\frac{dx}{dt} = 2$ units/s

slope of the curve $= \frac{dy}{dx} = 5 - 6x^2 = m$

$\frac{dm}{dt} = -12x\frac{dx}{dt} = -12x(2) = -24x$

at $x = 2, \frac{dm}{dt} = -24(2) = -48$

Hence, slope of curve is decreasing at the rate of 48

b. Area/perimeter/circumference of circle:

1.

If the circumference of circle is increasing at the constant rate, prove that rate of change of area of circle is directly proportional to its radius.

Sol.

Let 'r' be the radius, C the circumference and A the area of the circle.

Then, $\frac{dC}{dt} = k$ (Constant), also $C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{k}{2\pi}$

$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi\frac{dr}{dt} = 2\pi r \cdot \frac{k}{2\pi} = kr$, $\therefore$ the rate of change of area is directly proportional to its radius

2.

The area of the circle is increasing at a uniform rate of 2 cm^2/sec. How fast is the circumference of the circle increasing when the radius r = 5 cm ?

Sol.

Let $C = 2\pi r$, be the circumference of the circle,

$$\frac{d\left(\pi r^2\right)}{dt} = 2 \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r} \text{cm / sec}$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = \frac{2}{r} = \frac{2}{5} \text{cm/sec at } r = 5\text{cm}$$

c. Area/perimeter/diagonal/sides of a rectangle :

1.

The area of an expanding rectangle is increasing at the rate of 48 cm^2/s. The length of the rectangle is always square of its breadth. At what rate the length of rectangle increasing at an instant, when breadth = 4.5 cm ?

Sol.

Let the length and breadth of the expanding rectangle at any time be 'x' and 'y' respectively.

Then, $x = y^2$, $A(\text{Area}) = xy = x^{\frac{3}{2}}$

$$\frac{dA}{dt} = \frac{3}{2}\sqrt{x}\,\frac{dx}{dt}$$

$$\Rightarrow 48 = \frac{3}{2}\sqrt{(4.5)^2}\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{64}{9}\text{ cm/s}$$

d. Area/perimeter/median/altitude/sides of a triangle :

1.

A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall ?

Sol.

$x^2 + y^2 = 169$

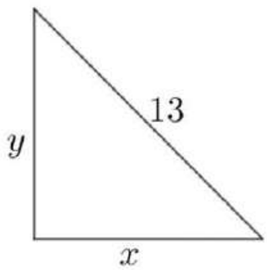
Differentiate both sides w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 12(2) + 5\left(\frac{dy}{dt}\right) = 0 [\because \text{ when } x = 12m, y = 5m]$$

$$\Rightarrow \frac{dy}{dt} = -\frac{24}{5}$$

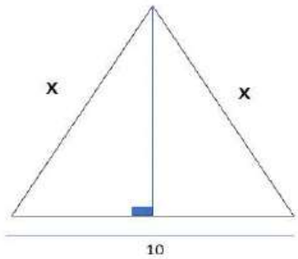
Hence, the height decreases at the rate of $\frac{24}{5}$ m/s



2.

If equal sides of an isosceles triangle with fixed base 10 cm are increasing at the rate of 4 cm/sec, how fast is the area of triangle increasing at an instant when all sides become equal ?

Sol.



Let the equal side be 'x', then $\frac{dx}{dt} = 4cm/s$

$$A = 5\sqrt{x^2 - 25} \Rightarrow \frac{dA}{dt} = \frac{5x}{\sqrt{x^2-25}}\frac{dx}{dt}$$

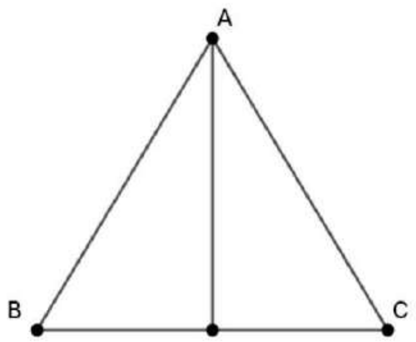
$$\left.\frac{dA}{dt}\right]_{x=10} = \frac{40}{\sqrt{3}} cm^2/s$$

3.

The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing.

Sol.

In an equilateral triangle, median is same as altitude. Let 'h' denote the length of the median (or altitude) and 'x' be the side of Δ ABC.



Then, $h = \frac{\sqrt{3}}{2}x$ or $x = \frac{2h}{\sqrt{3}}$ ________ (i)

It is given that $\frac{dh}{dt} = 2\sqrt{3}$ So, by (i) we have

4.

The side of an equilateral triangle is increasing at the rate of 3 cm/s. At what rate its area increasing when the side of the triangle is 15 cm ?

Sol.

Let 'a' be the side of the triangle, so $\frac{da}{dt} = 3\text{ cm/s}$

Now area of an equilateral triangle, $A = \frac{\sqrt{3}}{4}a^2$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}a}{2} \times \frac{da}{dt}$$

$$\therefore \left.\frac{dA}{dt}\right]_{a=15\text{cm}} = \frac{\sqrt{3}\times 15}{2} \times 3 = \frac{45\sqrt{3}}{2}\text{ cm}^2/\text{s}$$

e. Cube volume/surface area changing:

1.

The volume of a cube is increasing at the rate of 6 cm^3/s. How fast is the surface area of cube increasing, when the length of an edge is 8 cm ?

Sol.

Given, $\frac{dV}{dt} = 6\text{ cm}^3/\text{sec}$. Since, $V = x^3$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 6 = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2}{x^2}\text{ cm/sec}$$

Now, Surface Area = S = $6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} = 3\text{ cm}^2/\text{sec}$

2.

The surface area of a cube increases at the rate of 72 cm^2/sec. Find the rate of change of its volume, when the edge of the cube measures 3 cm.

Sol.

Let edge of cube be 'x cm'

$$S = 6x^2, \frac{dS}{dt} = 72cm^2/sec \Rightarrow 12x\frac{dx}{dt} = 72 \Rightarrow \frac{dx}{dt} = \frac{6}{x}$$

$$\text{Volume, } V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3 \times x^2 \times \frac{6}{x} = 18x, \left.\frac{dV}{dt}\right]_{x=3} = 54cm^3/sec$$

$\therefore$ Volume is increasing at the rate of 54 cm^3/sec

f. Volume/Surface Area of Sphere :

1.

The radius of an air bubble is increasing at the rate of 0·5 cm/s. At what rate is the surface area of the bubble increasing when the radius is 1·5 cm ?

Sol.

$$\frac{dr}{dt} = 0.5\,\text{cm/s} \quad (\text{given})$$

$$\text{Now}, S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dt} = 8\pi r.\frac{dr}{dt}$$

$$\therefore \left.\frac{dS}{dt}\right]_{r=1.5\text{ cm}} = 8\pi(1.5)(0.5) = 6\pi\,\text{cm}^2/\text{s}$$

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2.

Surface area of a balloon (spherical), when air is blown into it, increases at a rate of 5 mm^2/s. When the radius of the balloon is 8 mm, find the rate at which the volume of the balloon is increasing.

Sol.

$\frac{dS}{dt} = 5\ mm^2/s, \quad \left(\frac{dV}{dt}\right)_{r=8} = ?$

$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r.\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{8\pi r}$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2.\frac{dr}{dt} \Rightarrow \frac{dV}{dt} = \frac{5}{2}r$

$\Rightarrow \left(\frac{dV}{dt}\right)_{r=8} = 20\ mm^3/s$

3.

Let the volume of a metallic hollow sphere be constant. If the inner radius increases at the rate of 2 cm/s, find the rate of increase of the outer radius when the radii are 2 cm and 4 cm respectively.

Sol.

$\frac{dr}{dt} = 2\ cm/s,, \quad \left(\frac{dR}{dt}\right)_{R=4, r=2} = ?$

$V = \frac{4}{3}\pi(R^3 - r^3) \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi(3R^2.\frac{dR}{dt} - 3r^2\frac{dr}{dt})$

When R = 4 cm and r = 2 cm,

$0 = \frac{4}{3}\pi[3(4)^2.\frac{dR}{dt} - 3(2)^2(2)]$

$\Rightarrow \frac{dR}{dt} = \frac{1}{2}\ cm/s$

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5.

A spherical medicine ball when dropped in water dissolves in such a way that the rate of decrease of volume at any instant is proportional to its surface area. Calculate the rate of decrease of its radius.

Sol.

Let 'V' and 'S' be the volume and surface area of the spherical medicine ball with radius 'r'.

$$\frac{dV}{dt} = -kS,\ k > 0$$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow -kS = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow -k\left(4\pi r^2\right) = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -k$$

$\therefore$ **Radius decreases at a constant rate.**

g. Volume/Surface Area of cylinder :

1.

The radius of a cylinder is decreasing at a rate of 2 cm/s and the altitude is increasing at the rate of 3 cm/s. Find the rate of change of volume of this cylinder when its radius is 4 cm and altitude is 6 cm.

Sol.

$$\frac{dr}{dt} = -2\ cm/s,\quad \frac{dh}{dt} = 3\ cm/s\ ,\ \left(\frac{dV}{dt}\right)_{r=4,h=6} = ?$$

$$V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi r.\frac{dr}{dt}.h + \pi r^2 \frac{dh}{dt}$$

When r = 4 cm and h = 6 cm,

$$\frac{dV}{dt} = 2\pi(4)(-2)(6) + \pi(4)^2(3) = -48\pi\ cm^3/s$$

Volume is decreasing at the rate of 48 π cm³/s

2.

A cylindrical water container has developed a leak at the bottom. The water is leaking at the rate of 5 cm^3/s from the leak. If the radius of the container is 15 cm, find the rate at which the height of water is decreasing inside the container, when the height of water is 2 metres.

Sol.

Let V, r, h be the volume, radius and height of cylindrical container.

Given $\frac{dV}{dt} = -5\,\text{cm}^3/\text{s}$

$V = \pi r^2 h = \pi(15)^2 h = 225\pi h$

$\therefore \frac{dV}{dt} = 225\pi \frac{dh}{dt} \Rightarrow -5 = 225\pi \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = -\frac{5}{225\pi} = -\frac{1}{45\pi}$

$\therefore$ Height of the water is decreasing at the rate of $\frac{1}{45\pi}$ cm/s

h. Volume/Surface Area of cone :

1.

Sand is pouring from a pipe at the rate of 15 cm^3/minute. The falling sand forms a cone on the ground such that the height of the cone is always one-third of the radius of the base. How fast is the height of the sand cone increasing at the instant when the height is 4 cm ?

Sol.

Let 'V' be the volume, 'h' and 'r' be the height and radius of the cone,

$$\therefore h = \frac{r}{3},\ \frac{dV}{dt} = 15\ cm^3/min$$

$$V = \frac{1}{3}\pi r^2 h = 3\pi h^3$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right) = 15 \Rightarrow 3\pi \frac{d\left(h^3\right)}{dt} = 15 \Rightarrow \left.\frac{dh}{dt}\right]_{h=4} = \left.\frac{5}{3\pi h^2}\right]_{h=4} = \frac{5}{48\pi}\ cm/min$$

6.2.Increasing/decreasing :

Show that/Prove that function increasing/decreasing :

a. Algebraic quadratic/cubic polynomial :

1.

Consider the statement "There exists at least one value of $b \in \mathbb{R}$ for which $f(x) = \frac{b}{x}$, $b \neq 0$ is strictly increasing in $\mathbb{R} - \{0\}$."

State True or False. Justify.

Sol.

The given statement is "True".

$f'(x) = -\frac{b}{x^2}$

for $b < 0$, $f'(x) > 0$ in $(-\infty, 0)$ and $(0, \infty)$

$\therefore$ f(x) is strictly increasing in both these intervals.

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2.

If $f(x) = x + \frac{1}{x}$, $x \geq 1$, show that f is an increasing function.

$$f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Now $\frac{x^2 - 1}{x^2} \geq 0$ for all $x \geq 1$

$\Rightarrow f'(x) \geq 0 \Rightarrow f$ is an increasing function.

b. Trigonometric:

1.

Prove that $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \dfrac{\pi}{2}\right]$.

Sol.

$$f(x) = \frac{4\sin x}{2+\cos x} - x \Rightarrow f'(x) = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$$\text{when } x \in [0, \tfrac{\pi}{2}],\ \cos x \geq 0 \Rightarrow \frac{\cos x(4-\cos x)}{(2+\cos x)^2} \geq 0$$

Thus, f is increasing on $[0, \frac{\pi}{2}]$.

2.

Show that the function $f(x) = \dfrac{16\sin x}{4+\cos x} - x$, is strictly decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$.

Ans

$$f'(x) = \frac{16[4+\cos x]\cos x + 16\sin^2 x}{(4+\cos x)^2} - 1$$

$$= \frac{\cos x\,(56 - \cos x)}{(4+\cos x)^2}$$

in $(\frac{\pi}{2}, \pi)$, $\cos x < 0 \Rightarrow f'(x) < 0$

$\therefore$ f(x) in strictly decreasing in $(\frac{\pi}{2}, \pi)$

3.

Show that the function f given by f(x) = $\sin x + \cos x$, is strictly decreasing in the interval $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$.

Sol.

$$f(x) = \sin x + \cos x \Rightarrow f'(x) = \cos x - \sin x$$

$f'(x) = 0 \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ Thus,in the interval $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$ $f'(x) < 0$

$\therefore f$ is strictly decreasing function on $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$

4.

Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left[0, \frac{\pi}{4}\right]$.

Sol.

$$\mathbf{f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)}$$

For $x \in \left[0, \frac{\pi}{4}\right]$, $\cos x \geq \sin x$

$\Rightarrow f'(x) \geq 0$ **, f is an increasing function in** $\left[0, \frac{\pi}{4}\right]$

5.

If $f(x) = a(\tan x - \cot x)$, where $a > 0$, then find whether f(x) is increasing or decreasing function in its domain.

Sol.

$f'(x) = a(\sec^2 x + \text{cosec}^2 x)$

As $a > 0$ and $\sec^2 x$, $\text{cosec}^2 x$ are squares, $f'(x) > 0$

$\therefore$ f(x) is an increasing function in its domain.

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c. Exponential:

1.

Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain.

Sol.

$$f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2}$$

$$= e^x + \frac{1}{e^x} + \frac{x^2}{1+x^2} > 0 \; for \; all \; x \in R$$

$\therefore$ f is strictly increasing over its domain R

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Finding increasing/decreasing intervals :

a. Algebraic quadratic/cubic/biquadratic polynomial function :

1.

Determine whether the function $f(x) = x^2 - 6x + 3$ is increasing or decreasing in [4, 6].

Sol.

$f(x) = x^2 - 6x + 3, \; x \in [4,6]$

We have $f'(x) = 2x - 6$

For all x such that $4 < x < 6, \; 2 < 2x - 6 < 6$

$\Rightarrow f'(x) > 0$ for all $x \in (4,6)$

Hence, f is increasing over [4, 6].

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2.

Find the interval in which the function f(x) = $2x^3 - 3x$ is strictly increasing.

Sol.

Here, $f(x) = 2x^3 - 3x$

$$f'(x) = 6x^2 - 3 = 3(2x^2 - 1)$$

Now, $f'(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

f(x) is strictly increasing in $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

3.

Find the interval in which the function $x^3 - 12x^2 + 36x + 17$ is strictly increasing.

Sol. $f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6)$

f is strictly increasing, $f'(x) > 0$

$3(x-2)(x-6) > 0 \Rightarrow x \in (-\infty, 2) \cup (6, \infty)$

4.

Find the intervals in which the function f(x) = $x^4 - 4x^3 + 4x^2 + 15$, is strictly increasing.

Sol.

$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$

$f'(x) = 0$ gives $x = 0, 1, 2$

for strictly increasing, $f'(x) > 0$

$x \in (0,1) \cup (2, \infty)$

5.

Find the interval in which the function $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing.

Sol.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$\Rightarrow 4x^2(x-3) < 0 \text{ for } x < 3, x \neq 0$$
$$\Rightarrow f'(x) < 0 \text{ for } x < 3, x \neq 0$$
Thus, $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing on $(-\infty, 0) \cup (0,3)$

OR

Thus, $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing on $(-\infty, 0] \cup [0,3]$

6.

Find the interval in which the function $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is decreasing.

Sol.

$$f'(x) = 3x^2 - \frac{3}{x^4}$$
$$f'(x) = 0 \Rightarrow \frac{3(x^6-1)}{x^4} = 0 \Rightarrow \frac{3(x^3-1)(x^3+1)}{x^4} = 0$$
$$\Rightarrow x = -1, 1 \qquad (\because x \neq 0)$$
$$\therefore f(x) \text{ is decreasing when } x \in [-1,1] - \{0\}$$

6.

Determine those values of x for which $f(x) = \frac{2}{x} - 5, x \neq 0$ is increasing or decreasing.

Sol.

$f'(x) = \frac{-2}{x^2} < 0$

Hence f is decreasing in its domain.

7.

Find the interval in which $f(x) = x + \frac{1}{x}$ is always increasing, $x \neq 0$.

Sol.

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

$f'(x) = 0$ gives $x = 1, -1$

f is decreasing in $(-1, 0) \cup (0, 1)$ as $f'(x) < 0$

f is increasing in $R - (-1, 1)$ as $f'(x) > 0$

8.

Find the intervals in which function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}$ is (i) increasing (ii) decreasing.

Sol.

$f(x) = 5x^{3/2} - 3x^{5/2} \Rightarrow f'(x) = \frac{15}{2}\sqrt{x}(1-x)$

For increasing / decreasing, put $f'(x) = 0$

$\Rightarrow x = 0, 1$

(i) When $x \in [0,1]$, $f'(x) \geq 0$. So, f is increasing when $x \in [0,1]$

(The intervals $(0,1)$, $[0,1)$ or $(0,1]$ can also be considered.)

(ii) When $x \in [1, \infty)$, $f'(x) \leq 0$. So, f is decreasing when $x \in [1, \infty)$

(The interval $(1, \infty)$ can also be considered.)

9.

Determine the values of x for which $f(x) = \frac{x-4}{x+1}, x \neq -1$ is an increasing or a decreasing function.

Sol.

$$f'(x) = \frac{x+1-x+4}{(x+1)^2} = \frac{5}{(x+1)^2} > 0$$

Hence f is increasing in its domain.

10.

Find the values of 'a' for which $f(x) = x^2 - 2ax + b$ is an increasing function for $x > 0$.

Sol.

$$f'(x) = 2x - 2a$$

$$0 < x < \infty \Rightarrow -2a < 2x - 2a < \infty \Rightarrow -2a < f'(x) < \infty$$

$f(x)$ **is increasing iff** $f'(x) \geq 0$

$$\Rightarrow -2a \in [0, \infty) \Rightarrow a \in (-\infty, 0] \text{ or } (-\infty, 0)$$

11.

Find the least value of 'a' so that $f(x) = 2x^2 - ax + 3$ is an increasing function on [2, 4].

Sol.

$$f(x) = 2x^2 - ax + 3 \Rightarrow f'(x) = 4x - a$$

Now $2 \leq x \leq 4 \Rightarrow 8 - a \leq 4x - a \leq 16 - a$

For f to be an increasing function, $f'(x) \geq 0$

$$\Rightarrow 8 - a \geq 0 \Rightarrow a \leq 8$$

$\therefore$ Least value of a does not exist.

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b.Trigonometric function:

1.

Find the intervals in which the function given by

$f(x) = \sin 3x,\ x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Sol.

$$f(x) = \sin 3x \Rightarrow f'(x) = 0 \Rightarrow 3\cos 3x = 0 \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$$

$$f'(x) \geq 0, \forall x \in \left[0, \frac{\pi}{6}\right] \Rightarrow f(x) \text{ is increasing on } \left[0, \frac{\pi}{6}\right]$$

$$f'(x) \leq 0, \forall x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow f(x) \text{ is decreasing on } \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$$

2.

Find the interval/intervals in which the function $f(x) = \sin 3x - \cos 3x$, $0 < x < \frac{\pi}{2}$ is strictly increasing.

Sol.

$$f'(x) = 3\cos 3x + 3\sin 3x$$

$$f'(x) = 0 \Rightarrow \sin 3x = -\cos 3x \Rightarrow x = \frac{\pi}{4}$$

For $x \in \left(0, \frac{\pi}{4}\right)$, $3\cos 3x + 3\sin 3x > 0$

$\Rightarrow f'(x) > 0$, f is strictly increasing function in $(0, \frac{\pi}{4})$ or $(0, \frac{\pi}{4}]$

3.

Find the values of ‘a’ for which $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing on $\mathbb{R}$.

Sol.

‘f ’ is a decreasing function iff $f'(x) \le 0 \Rightarrow \sqrt{3}\cos x + \sin x - 2a \le 0$

$$\Rightarrow \sin\left(x + \frac{\pi}{3}\right) \le a$$

We know that for all $x \in R, -1 \le \sin\left(x + \frac{\pi}{3}\right) \le 1$

$$\Rightarrow a \in [1, \infty) \text{ or } (1, \infty)$$

OR

$$f'(x) = 3\cos 3x + 3\sin 3x$$

$$f'(x) = 0 \Rightarrow \sin 3x = -\cos 3x \Rightarrow x = \frac{\pi}{4}$$

For $x \in \left(0, \frac{\pi}{4}\right)$, $3\cos 3x + 3\sin 3x > 0$

$\Rightarrow f'(x) > 0$, **f is strictly increasing function in** $(0, \frac{\pi}{4})$ **or** $(0, \frac{\pi}{4}]$

4.

Find the values of ‘a’ for which $f(x) = \sin x - ax + b$ is increasing on R.

Sol.

$$f'(x) = \cos x - a$$

For f(x) to be increasing, $f'(x) \ge 0$

$i.e., \cos x \ge a$

Since, $-1 \le \cos x \le 1$

$\Rightarrow a \le -1$

Hence, $a \in (-\infty, -1]$. (Also, accept $a \in (-\infty, -1)$)

d. Logarithmic function:

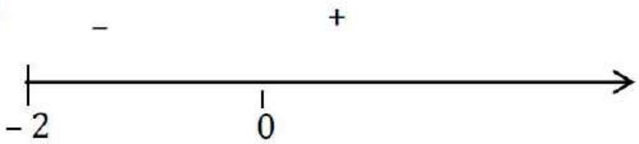
1.

Find the sub-intervals in which $f(x) = \log(2 + x) - \dfrac{x}{2+x}$, $x > -2$ is increasing or decreasing.

Sol.

$$f'(x) = \frac{1}{2+x} - \frac{2}{(2+x)^2} = \frac{x}{(2+x)^2}$$

Sign of $f'(x)$



f(x) is decreasing in $(-2, 0)$
and increasing in $(0, \infty)$

2.

Find the intervals in which the function $f(x) = \dfrac{\log x}{x}$ is strictly increasing or strictly decreasing.

Sol.

$$f(x) = \frac{\log x}{x} \Rightarrow f'(x) = \frac{1 - \log x}{x^2} \; ; x > 0$$

for strictly increasing/decreasing, put $f'(x) = 0 \Rightarrow x = e$

for strictly increasing, $x \in (0, e)$ and for strictly decreasing $x \in (e, \infty)$

6.3 Maximum/minimum value :

a. Polynomial/algebraic/log function :

If Intervals are not given :

1.

Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Sol.

$f'(x) = 12x^2 - 36x + 27$

$= 3(2x - 3)^2 \geq 0$ for all $x \in R$

$\therefore$ f is increasing on R .

Hence f(x) does not have maxima or minima.

2.

Find the local maxima and local minima of the function $f(x) = \frac{8}{3}x^3 - 12x^2 + 18x + 5.$

Sol.

$f(x) = \frac{8}{3}x^3 - 12x^2 + 18x + 5$

$\Rightarrow f'(x) = 8x^2 - 24x + 18$

$= 2(4x^2 - 12x + 9) = 2(2x - 3)^2$

For critical points, Put $f(x) = 0$

$\Rightarrow 2(2x - 3)^2 = 0 \Rightarrow x = \frac{3}{2}$

since $f'(x)$ does not change the sign as crosses $x = \frac{3}{2}$ from left to right,

f has no local maxima or local minima.

3.

Find the maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 30$.

Sol.

$y = -x^3 + 3x^2 + 9x - 30$

Slope of the curve, $m = \frac{dy}{dx} = -3x^2 + 6x + 9$

$\Rightarrow \frac{dm}{dx} = -6x + 6$

For maximum/ minimum slope, put $\frac{dm}{dx} = 0$

$\Rightarrow x = 1$

As $\frac{d^2m}{dx^2} = -6 < 0 \therefore m$ is maximum at $x = 1$

Maximum slope $= -3(1)^2 + 6(1) + 9 = 12$

4.

If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}$ $(x \neq 0)$ respectively, find the value of (M – m).

Sol.

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}$

$f'(x) = 0 \Rightarrow x = -1,\ 1$

$f''(x) = \frac{2}{x^3} \Rightarrow f''(-1) = -2 < 0$

$\therefore$ -1 is a point of local maximum

The local maximum value $= f(-1) = -2 = M$

$f''(1) = 2 > 0$

$\therefore$ 1 is point of local minimum

The local minimum value $= f(1) = 2 = m$

$M - m = -4$

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5.

Find local maximum value and local minimum value (whichever exists) for the function $f(x) = 4x^2 + \frac{1}{x}$ $(x \neq 0)$.

Sol.

$f(x) = 4x^2 + \frac{1}{x}$ $(x \neq 0)$

$f'(x) = 8x - \frac{1}{x^2} = 0$

$\Rightarrow x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2}$

$f''(x) = 8 + \frac{2}{x^3} > 0$ at $x = \frac{1}{2}$

$\therefore$ Local minimum value $= f\left(\frac{1}{2}\right) = 3$

6.

The relation between the height of the plant (y cm) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to sunlight.

(i) Find the rate of growth of the plant with respect to sunlight.

(ii) In how many days will the plant attain its maximum height ? What is the maximum height ?

Sol.

$(i)\, y = 4x - \frac{1}{2}x^2 \Rightarrow \frac{dy}{dx} = (4 - x)$ cm/day

(ii) For maximum height, $\frac{dy}{dx} = 0 \Rightarrow x = 4$ days

as $\frac{d^2y}{dx^2} < 0$, number of days $= 4$

Now, Maximum height $= y(4) = 16 - \frac{1}{2}(16) = 8$ cm

7.

It is given that function $f(x) = x^4 - 62x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of 'a', hence obtain all other points where the given function f(x) attains local maximum or local minimum values.

Sol.

$f(x) = x^4 - 62x^2 + ax + 9 \Rightarrow f'(x) = 4x^3 - 124x + a$

as at $x = 1$, f attains local maximum value, $f'(1) = 0 \Rightarrow a = 120$

now, $f'(x) = 4x^3 - 124x + 120 = 4(x-1)(x^2 + x - 30) = 4(x-1)(x-5)(x+6)$

Critical points are $x = -6, 1, 5$

$f''(x) = 12x^2 - 124$

$f''(-6) > 0$, $f''(1) < 0$, $f''(5) > 0$

so f attains local maximum value at $x = 1$ and local minimum value at $x = -6, 5$

8.

Given that $f(x) = \dfrac{\log x}{x}$, find the point of local maximum of f(x).

Sol.

$\mathbf{f'(x) = \dfrac{1 - \log x}{x^2}, \therefore f'(x) = 0 \Rightarrow \log x = 1 \Rightarrow x = e}$

$\mathbf{f''(x) = \dfrac{2x \log x - 3x}{x^4} \Rightarrow f''(e) = -\dfrac{1}{e^3} < 0}$ **i.e. $x = e$ is a point of local maximum.**


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If Intervals are given :

1.

Find the absolute maximum and absolute minimum of function $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $[1, 5]$.

Sol

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$\Rightarrow f'(x) = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$$

$$f'(x) = 0 \Rightarrow x = 2, 3 \in [1, 5]$$

Now $f(1) = 24, f(2) = 29, f(3) = 28, f(5) = 56$

Hence, the absolute maximum value is 56 and the absolute minimum value is 24.

2.

Find the absolute maximum and minimum values of the function $f(x) = 12x^{4/3} - 6x^{1/3}, x \in [0, 1]$.

Sol.

$$f'(x) = 16\, x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}}$$

For critical points, $f'(x) = 0$

$$\Rightarrow 16\,x = 2 \Rightarrow x = \frac{1}{8}$$

x	f(x)
0	0
$\frac{1}{8}$	$\frac{-9}{4}$ (Absolute minimum)
1	6 (Absolute maximum)


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3.

Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \frac{x}{2} + \frac{2}{x}$, on the interval [1, 2].

Sol.

$$f(x) = \frac{x}{2} + \frac{2}{x} \quad ; \; x \in [1,2]$$

$$\Rightarrow f^{1}(x) = \frac{1}{2} - \frac{2}{x^2}$$

for absolute maximum / minimum, put $f'(x) = 0$

$$\Rightarrow x^2 = 4 \Rightarrow x = 2$$

Now, $f(1) = \frac{5}{2}$ and $f(2) = 2$

$\therefore$ absolute maximum value $= \frac{5}{2}$ and absolute minimum value $= 2$

prepared by : **BALAJI KANCHI**

b. Trigonometric function:

1.

Prove that $f(x) = \sin x + \sqrt{3} \cos x$ has maximum value at $x = \frac{\pi}{6}$.

Sol.

$f'(x) = \cos x - \sqrt{3} \sin x$

Clearly $f'(\frac{\pi}{6}) = 0$

$f''(x) = -\sin x - \sqrt{3} \cos$

Clearly $f''(\frac{\pi}{6}) < 0$

2.

Find the maximum and minimum values of the function given by $f(x) = 5 + \sin 2x$.

Sol.

$$\text{We know that } -1 \le \sin 2x \le 1$$

$$\Rightarrow 4 \le \sin 2x + 5 \le 1 + 5$$

$$\Rightarrow 4 \le \sin 2x + 5 \le 6$$

So, maximum value is 6 and minimum value is 4

c. Max/min/least - Area/perimeter :

1.

Show that of all the rectangles with a fixed perimeter, the square has the greatest area.

Sol.

Let P be the perimeter of the rectangle, which is a constant. Also assume 'x' and 'y' be the length and breadth of the rectangle, then

$$2(x+y) = P \text{ and } A(\text{Area}) = xy = \frac{x}{2}(P-2x) = \frac{1}{2}(Px - 2x^2)$$

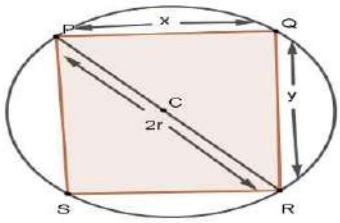
$$A'(x) = \frac{1}{2}(P-4x), \ \therefore \ A'(x) = 0 \Rightarrow x = \frac{P}{4}, y = \frac{P}{4}$$

$$A''(x) = -2 < 0 \text{ at } x = \frac{P}{4}, \ \therefore \text{Area of the rectangle is max. if it is a square.}$$

2.

.Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Sol.



Let the rectangle with sides x and y has maximum area

$\therefore A = x \cdot y$ where $x^2 + y^2 = 4r^2$, r being the radius

$$\therefore A = x\sqrt{4r^2 - x^2}$$

or $Z = x^2(4r^2 - x^2) = 4r^2x^2 - x^4$, where $Z = A^2$

$$\frac{dZ}{dx} = 8r^2x - 4x^3, \quad \frac{dZ}{dx} = 0 \Rightarrow x = \sqrt{2}r$$

$$\therefore y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

$$\frac{d^2Z}{dx^2} = 8r^2 - 12x^2 < 0 \text{ for } x^2 = 2r$$

$$\therefore x = y = \sqrt{2}r$$

Therefore, of all the rectangles inscribed in a given circle, the square has the maximum area.

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d. Sum/product of two numbers max/min :

1.

Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers.

Sol.

Let the two numbers be x and y. Then, x + y = 5 or y = 5 – x

Let S denote the sum of the cubes of these numbers. Then

$S = x^3 + y^3 = x^3 + (5 - x)^3$

$\frac{dS}{dx} = 3x^2 - 3(5 - x)^2 = 15(2x - 5)$

Now $\frac{dS}{dx} = 0$, gives $x = \frac{5}{2}$

Showing S is minimum at $x = \frac{5}{2}$

So, the two numbers are $\frac{5}{2}$ and $\frac{5}{2}$

$\Rightarrow \boldsymbol{x^2 + y^2 = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}}$

2.

If the product of two positive numbers is 9, find the numbers so that the sum of their squares is minimum.

Sol.

Let numbers be x and $\frac{9}{x}$ $\therefore$ required sum = $x^2 + \frac{81}{x^2}$ = f(x) say

$f'(x) = 2x - \frac{162}{x^3}$

$f'(x) = 0 \Rightarrow x = 3$

showing f(x) is minimum at x = 3

$\therefore$ sum is minimum when both numbers are 3

2.

Amongst all pairs of positive integers with product as 289, find which of the two numbers add up to the least.

Sol.

Let numbers be 'x' and 'y' such that $xy = 289 \Rightarrow y = \frac{289}{x}$, 'S' be their sum, then

$$S = x + y = x + \frac{289}{x}$$

$\frac{dS}{dx} = 1 - \frac{289}{x^2}, \frac{dS}{dx} = 0 \Rightarrow x = 17$, **a positive integer**

$$\left.\frac{d^2S}{dx^2}\right]_{x=17} = 289\left.\left(\frac{2}{x^3}\right)\right]_{x=17} > 0, \therefore S \text{ is minimum when } x = 17, y = 17$$

f. maximu/minimum/greatest/largest/least Volume :

1.

The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum.

Sol.

Let length of rectangle be x cm and breadth be $(150-x)$ cm.

Let r be the radius of cylinder $\Rightarrow 2\pi r = x \Rightarrow r = \dfrac{x}{2\pi}$

$$V = \pi r^2 h = \pi\left(\frac{x^2}{4\pi^2}\right)(150-x) = \frac{75x^2}{2\pi} - \frac{x^3}{4\pi}$$

$$\frac{dV}{dx} = \frac{150x}{2\pi} - \frac{3x^2}{4\pi}$$

$$\frac{dV}{dx} = 0 \Rightarrow x = 100\,\text{cm}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=100\text{ cm}} = -\frac{75}{\pi} < 0 \Rightarrow V \text{ is maximum when } x = 100\,\text{cm}.$$

Length of rectangle is 100 cm and breadth of rectangle is 50 cm.

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2.

Find dimensions of a rectangle of perimeter 12 cm which will generate maximum volume when swept along a circular rotation keeping the shorter side fixed as the axis.

Sol.

Let Length and Breadth of the rectangle be 'x' and 'y' respectively. Also 'r' be the radius of the cylinder then,

$$2(x+y)=12 \Rightarrow x+y=6\,,\ 2\pi r = x$$

$$\text{V(Volume of cylinder)} = \pi r^2 y \Rightarrow V = \pi\left(\frac{x}{2\pi}\right)^2(6-x) = \frac{1}{4\pi}\left(6x^2 - x^3\right)$$

$$V'(x) = 0 \Rightarrow \frac{1}{4\pi}\left(12x - 3x^2\right) = 0 \Rightarrow x = 4,\ (\because x \neq 0)$$

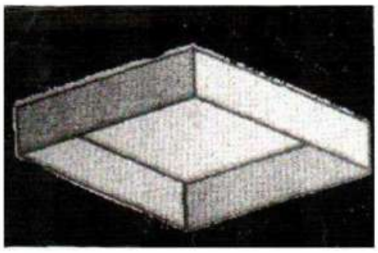
$$V''(x) = \frac{1}{4\pi}(12-6x) \Rightarrow V''(4) = -\frac{3}{\pi} < 0$$

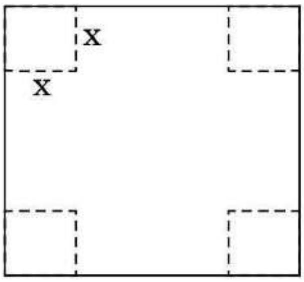
The volume of the cylinder obtained by the rotation will be maximum if the dimensions of the rectangle are $x = 4$ cm, $y = 2$ cm.

Case study :

1.

A factory makes an open cardboard box for a jewellery shop from a square sheet of side 18 cm by cutting off squares from each corner and folding up the flaps.





Based on the above information, answer ***any four*** of the following ***five*** questions, if x is the length of each square cut from corners : *4×1=4*

(i) The volume of the open box is :

(A) $4x(x^2 - 18x + 81)$

(B) $2x(2x^2 + 36x + 162)$

(C) $2x(2x^2 + 36x - 162)$

(D) $4x(x^2 + 18x + 81)$

(ii) The condition for the volume (V) to be maximum is :

(A) $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

(B) $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} > 0$

(C) $\frac{dV}{dx} > 0$ and $\frac{d^2V}{dx^2} = 0$

(D) $\frac{dV}{dx} < 0$ and $\frac{d^2V}{dx^2} = 0$

(iii) What should be the side of square to be cut off so that the volume is maximum ?

(A) 6 cm

(B) 9 cm

(C) 3 cm

(D) 4 cm

(iv) Maximum volume of the open box is :

(A) 423 cm^3

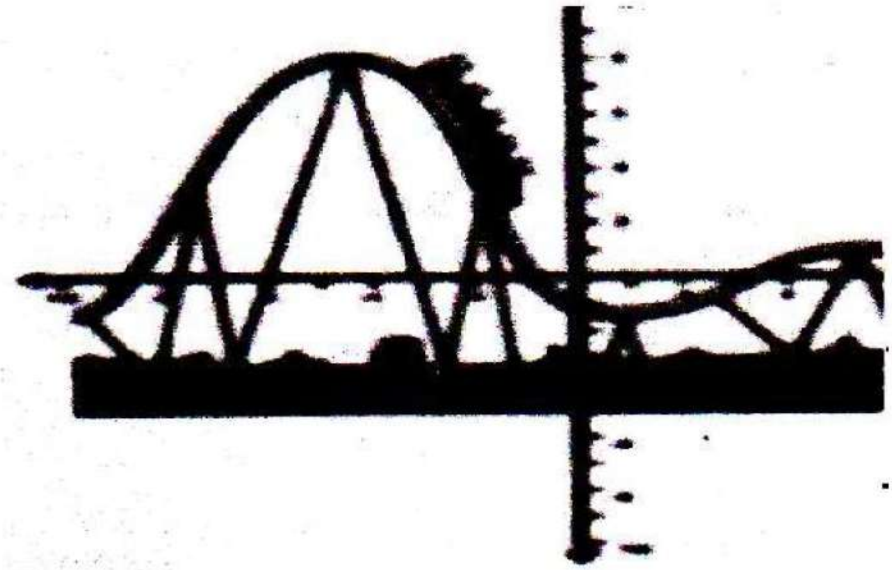
(B) 432 cm^3

(C) 400 cm^3

(D) 216 cm^3

2.

The equation of the path traced by a roller-coaster is given by the polynomial $f(x) = a(x + 9)(x + 1)(x - 3)$. If the roller-coaster crosses y-axis at a point $(0, -1)$, answer the following :



(i) Find the value of 'a'. 2

(ii) Find $f''(x)$ at $x = 1$. 2

Sol.

(i) $-1 = a(-27) \quad \Rightarrow a = \frac{1}{27}$

(ii) $f(x) = \frac{1}{27}(x + 9)(x + 1)(x - 3)$

$= \frac{1}{27}(x^3 + 7x^2 - 21x - 27)$

$f'(x) = \frac{1}{27}(3x^2 + 14x - 21)$

$f''(x) = \frac{6x + 14}{27}$

$f''(1) = \frac{20}{27}$

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3.

A tank, as shown in the figure below, formed using a combination of a cylinder and a cone, offers better drainage as compared to a flat bottomed tank.

A tap is connected to such a tank whose conical part is full of water. Water is dripping out from a tap at the bottom at the uniform rate of $2 \text{ cm}^3/\text{s}$. The semi-vertical angle of the conical tank is $45°$.

On the basis of given information, answer the following questions :

(i) Find the volume of water in the tank in terms of its radius r.

(ii) Find rate of change of radius at an instant when $r = 2\sqrt{2}$ cm.

(iii) (a) Find the rate at which the wet surface of the conical tank is decreasing at an instant when radius $r = 2\sqrt{2}$ cm.

OR

(iii) (b) Find the rate of change of height 'h' at an instant when slant height is 4 cm.

Sol.

((i) $v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$ [as $\theta = 45°$ gives $r = h$]

(ii) $\frac{dv}{dt} = \pi r^2 \frac{dr}{dt}$

$\Rightarrow \left(\frac{dr}{dt}\right)_{r=2\sqrt{2}} = -\frac{1}{4\pi}$ cm/sec

(iii)(a) $C = \pi r l = \pi r \sqrt{2}\, r = \sqrt{2}\, \pi r^2$

$\frac{dC}{dt} = \sqrt{2}\,\pi\, 2r \frac{dr}{dt}$

$\left(\frac{dC}{dt}\right)_{r=2\sqrt{2}} = -2$ cm^2/sec

OR

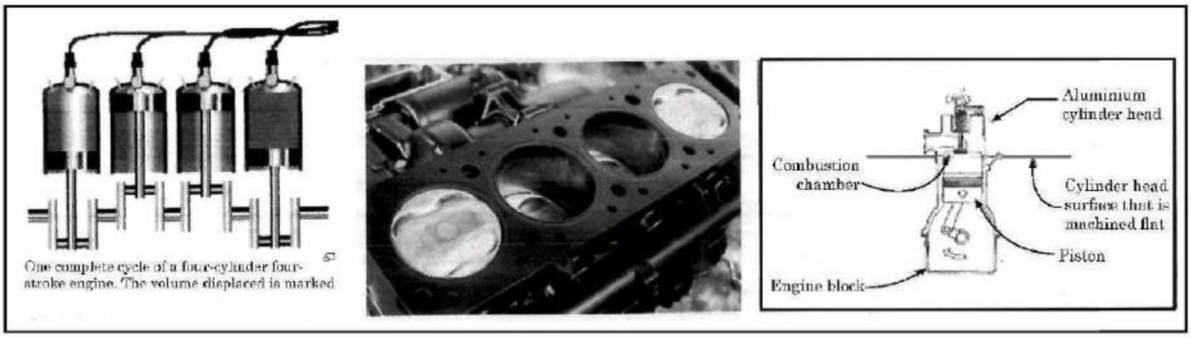
(iii)(b) $l^2 = h^2 + r^2$

$l = 4 \Rightarrow r = h = 2\sqrt{2}$

$h = r \Rightarrow \frac{dh}{dt} = \frac{dr}{dt} = -\frac{1}{4\pi}$ cm/sec

4.

Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore



The cylinder bore in the form of circular cylinder open at the top is to be made from a metal sheet of area 75π cm^2.

Based on the above information, answer the following questions :

(i) If the radius of cylinder is r cm and height is h cm, then write the volume V of cylinder in terms of radius r.

(ii) Find $\frac{dV}{dr}$.

(iii) (a) Find the radius of cylinder when its volume is maximum.

OR

(b) For maximum volume, h > r. State true or false and justify.

Sol.

(i) $\pi r^2 + 2\pi rh = 75\pi \Rightarrow h = \frac{75 - r^2}{2r}$, $\therefore V = \pi r^2 h = \frac{\pi}{2}\left(75r - r^3\right)$

(ii) $\frac{dV}{dr} = \frac{\pi}{2}\left(75 - 3r^2\right)$

(iii) $\frac{dV}{dr} = 0 \Rightarrow r = 5, \left.\frac{d^2V}{dr^2}\right]_{r=5} = \frac{\pi}{2}(-6r) < 0 \therefore$ volume is maximum when $r = 5$

Or

False,

$\frac{dV}{dr} = 0 \Rightarrow r = 5, \left.\frac{d^2V}{dr^2}\right]_{r=5} = \frac{\pi}{2}(-6r) < 0 \therefore$ volume is maximum when $r = 5$

As volume is maximum at $r = 5 \Rightarrow h = \frac{75 - 5^2}{2(5)} = 5 \Rightarrow h = r$

5.

The use of electric vehicles will curb air pollution in the long run.

The use of electric vehicles is increasing every year and estimated electric vehicles in use at any time t is given by the function V :

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$$

where t represents the time and t = 1, 2, 3.... corresponds to year 2001, 2002, 2003, ... respectively.

Based on the above information, answer the following questions :

(i) Can the above function be used to estimate number of vehicles in the year 2000 ? Justify.

(ii) Prove that the function V(t) is an increasing function.

Sol.

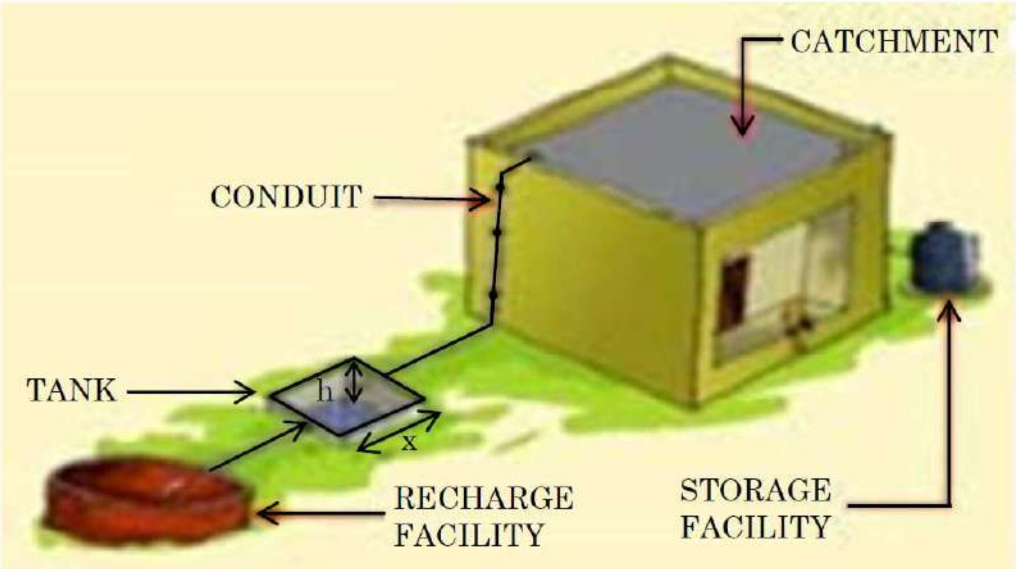
(i) For the year 2000, $t = 0$ & $V(0) = -2$ and the number of vehicles cannot be negative $\therefore$ the given function V(t) cannot be used.

(ii) $V'(t) = \frac{3}{5}t^2 - 5t + 25 = \frac{3}{5}\left[\left(t - \frac{25}{6}\right)^2 + \frac{875}{36}\right] > 0$, **$\therefore$ V(t) is an increasing function.**

6.

In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of 250 m^3. The cost of land is ₹ 5,000 per square metre and cost of digging increases with depth and for the whole tank, it is ₹ 40,000 h^2, where h is the depth of the tank in metres. x is the side of the square base of the tank in metres.

ELEMENTS OF A TYPICAL RAIN WATER HARVESTING SYSTEM



Based on the above information, answer the following questions :

(i) Find the total cost C of digging the tank in terms of x.

(ii) Find $\frac{dC}{dx}$.

(iii) (a) Find the value of x for which cost C is minimum.

OR

(iii) (b) Check whether the cost function C(x) expressed in terms of x is increasing or not, where $x > 0$.

Sol.

Ans(i)

(i) $C = 40000h^2 + 5000x^2$

as $x^2h = 250$

$$\Rightarrow C = \frac{40000\,(250)^2}{x^4} + 5000x^2$$

Ans(ii)

$$\text{(ii)}\ \frac{dC}{dx} = \frac{-160000\,(250)^2}{x^5} + 10000x$$

Ans(iii)

(iii)(a) For minimum cost $\frac{dC}{dx} = 0$

$$\Rightarrow 10000x^6 = 250 \times 250 \times 160000$$

$$\Rightarrow x = 10$$

showing $\frac{d^2C}{dx^2} > 0$ at $x = 10$

$\therefore$ cost is minimum when $x = 10$

OR

Ans(iii)

$$\text{(iii)(b)}\frac{dC}{dx} = \frac{-160000\,(250)^2}{x^4} + 10000x$$

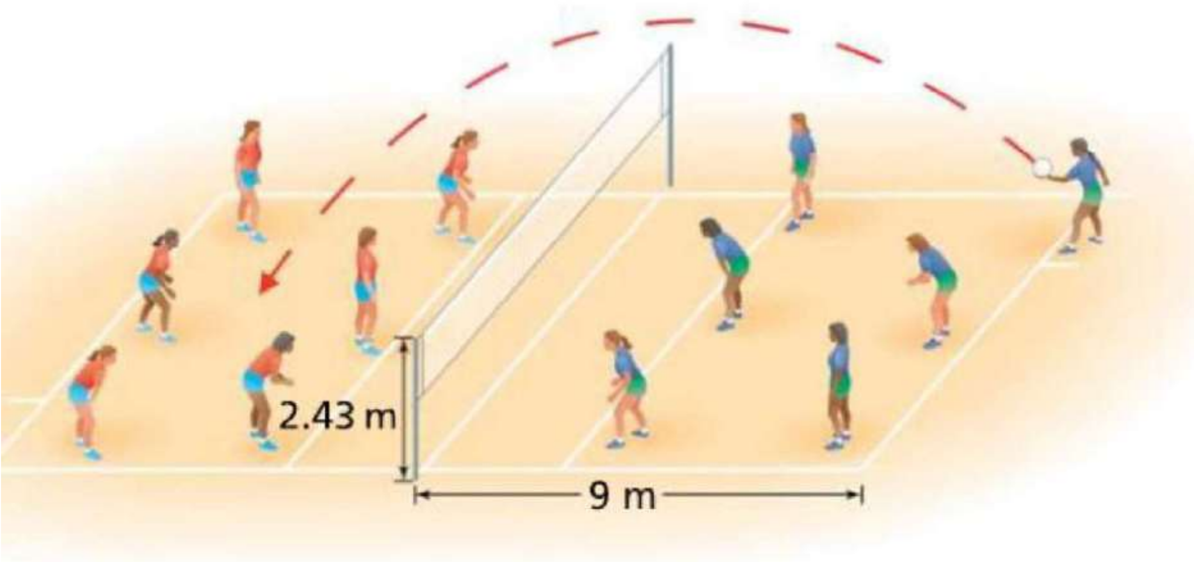
$\frac{dC}{dx} = 0$ gives $x = 10$

$\frac{dC}{dx} > 0$ in $(10, \infty)$ and $\frac{dC}{dx} < 0$ in $(0, 10)$.

Hence, cost function is neither increasing nor decreasing for $x > 0$

7.

A volleyball player serves the ball which takes a parabolic path given by the equation $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, where h(t) is the height of ball at any time t (in seconds), $(t \geq 0)$.



Based on the above information, answer the following questions :

(i) Is h(t) a continuous function ? Justify.

(ii) Find the time at which the height of the ball is maximum.

Sol.

Ans(i)

(i)$h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$

Clearly h(t) is a polynomial function, hence continuous.

Hence h(t) is a continuous function.

Ans(ii)

(ii)For maximum height ,

$\frac{dh}{dt} = 0 \Rightarrow -7t + \frac{13}{2} = 0$

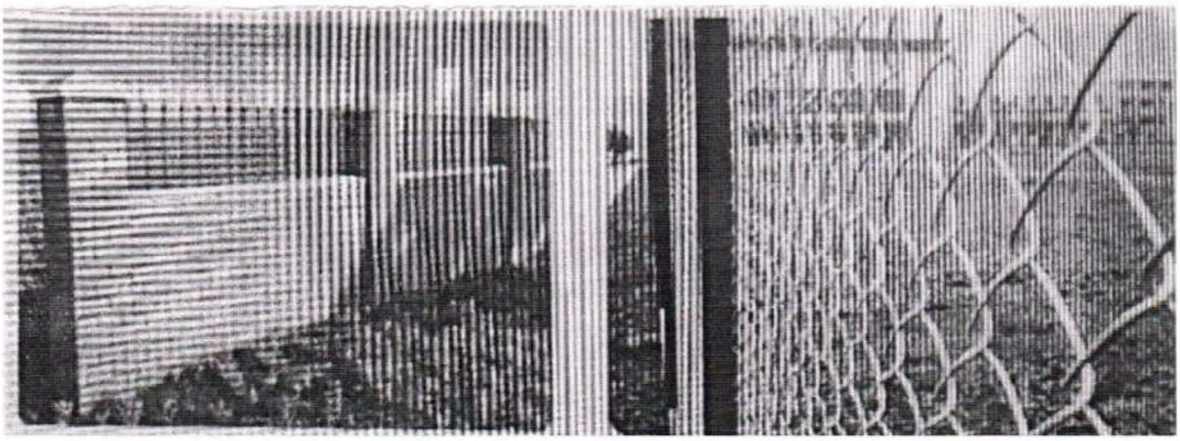
$\Rightarrow t = \frac{13}{14}$

$\frac{d^2h}{dt^2} = -7 < 0 \quad \therefore$ height is maximum at $t = \frac{13}{14}$

prepared by : **BALAJI KANCHI**

8.

Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.



Based on the above information, answer the following questions :

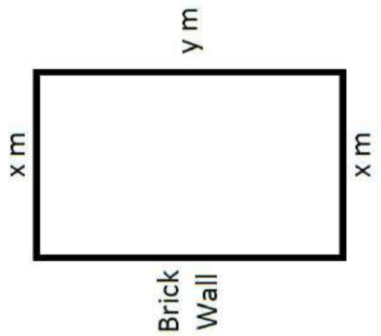
(i) Let 'x' metres denote the length of the side of the garden perpendicular to the brick wall and 'y' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write A(x), the area of the garden.

(ii) Determine the maximum value of A(x).

Sol.

(i) (a) $2x + y = 200$

(b) $A(x) = xy = x(200-x)$



(ii) From (a) and (b) of (1) we have

$A(x) = x(200 - 2x)$

$= 200x - 2x^2$

From max./min of A(x)

$\frac{dA}{dx} = 0$ i.e. $200 - 4x = 0$

$\Rightarrow x = 50.$

$\frac{d^2A}{dx^2} = -4$

$\left(\frac{d^2A}{dx^2}\right)_{x=50} < 0$

Hence, A(x) is maximum at $x = 50$

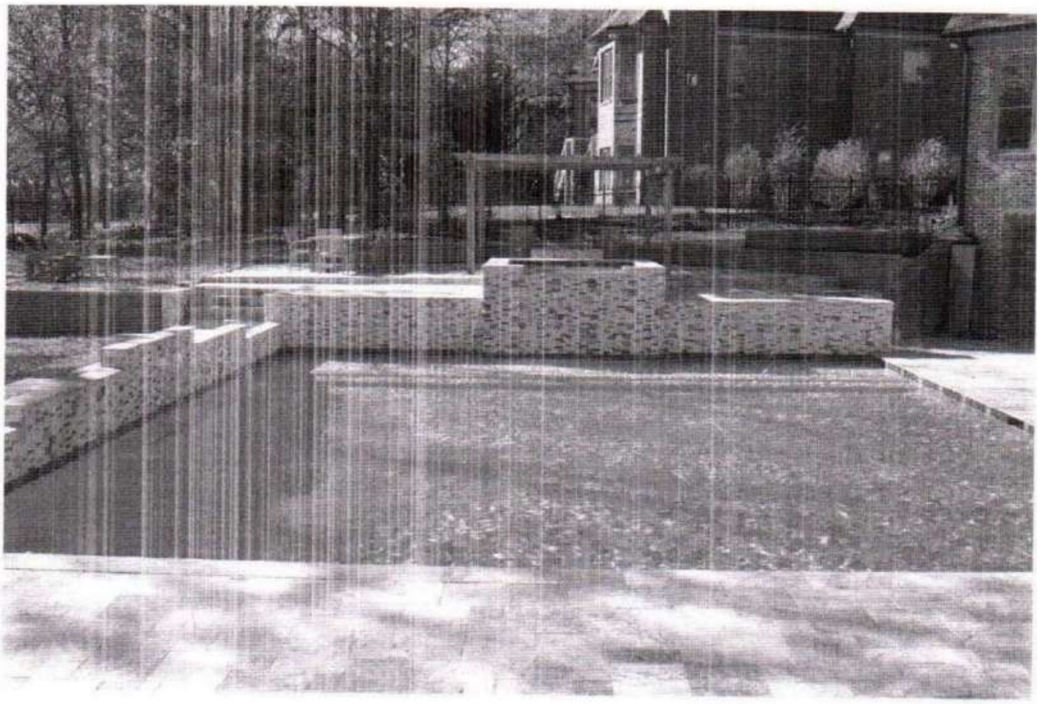
Thus, Max $A(x) = 200(50) - 2(50)^2$

$= 10000 - 5000$

$= 5000$ sqm.

9.

A housing society wants to commission a swimming pool for its residents. For this, they have to purchase a square piece of land and dig this to such a depth that its capacity is 250 cubic metres. Cost of land is ₹ 500 per square metre. The cost of digging increases with the depth and cost for the whole pool is ₹ 4000 $(\text{depth})^2$.



Suppose the side of the square plot is x metres and depth is h metres.
On the basis of the above information, answer the following questions :

(i) Write cost C(h) as a function in terms of h.

(ii) Find critical point.

(iii) (a) Use second derivative test to find the value of h for which cost of constructing the pool is minimum. What is the minimum cost of construction of the pool ?

OR

(iii) (b) Use first derivative test to find the depth of the pool so that cost of construction is minimum. Also, find relation between x and h for minimum cost.

Sol.

(i) Capacity = area × depth $= x^2h = 250 \Rightarrow x^2 = \frac{250}{h}$

$C\,(\text{cost}) = 500x^2 + 4000h^2$

$\Rightarrow C = 500\left(\frac{250}{h}\right) + 4000h^2 = \frac{125000}{h} + 4000h^2$

$(ii)\ \frac{dC}{dh} = -\frac{125000}{h^2} + 8000h$

$\frac{dC}{dh} = 0 \Rightarrow h = \frac{5}{2}m$ or $2.5m$

$(iii)(a)\ \frac{d^2C}{dh^2} = -125000\left(\frac{-2}{h^3}\right) + 8000 = \frac{250000}{h^3} + 8000$

$\left.\frac{d^2C}{dh^2}\right|_{h=2.5m} > 0 \Rightarrow$ Cost is minimum when $h = 2.5m$

Minimum cost $= C = \frac{125000}{\left(\frac{5}{2}\right)} + 4000\left(\frac{5}{2}\right)^2 = Rs.75{,}000$

OR

$(iii)(b)$ we already have found above that $h = \frac{5}{2}m$ when $\frac{dC}{dh} = 0$

for the values of h less than $\frac{5}{2}$ and close to $\frac{5}{2}$, $\frac{dC}{dh} < 0$

and, for the values of h more than $\frac{5}{2}$ and close to $\frac{5}{2}$, $\frac{dC}{dh} > 0$

By first derivative test, there is a minimum at $h = \frac{5}{2}$

Now, $x^2 = \frac{250}{h} \Rightarrow x^2 = \frac{250}{\left(\frac{5}{2}\right)} = 100 \Rightarrow x = 10m$

also, $x = 4h$

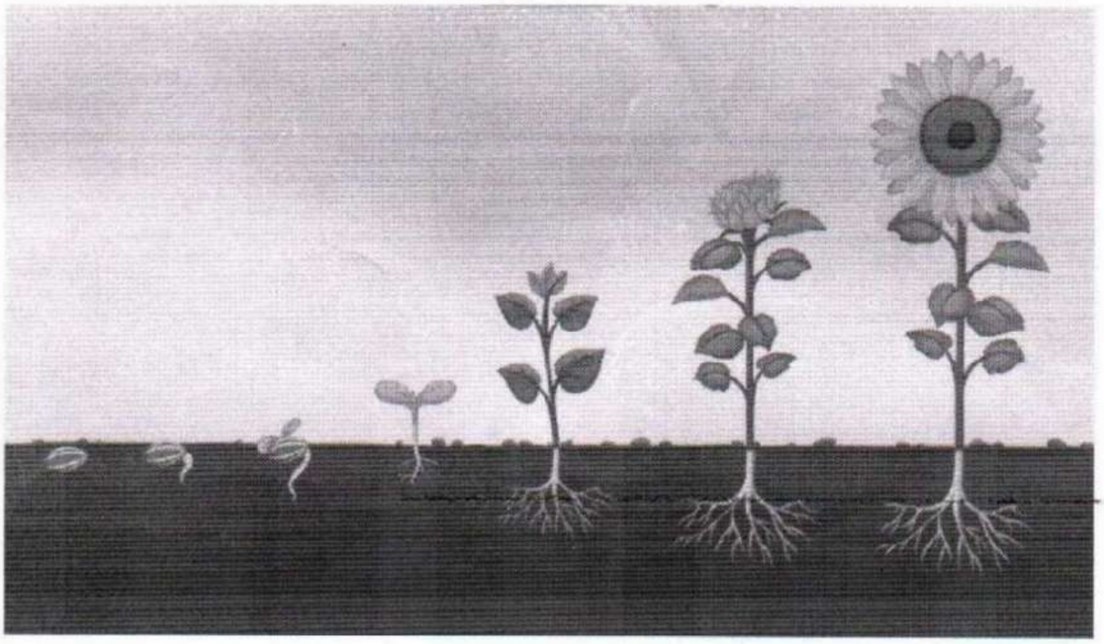
10.

In an agricultural institute, scientists do experiments with varieties of seeds to grow them in different environments to produce healthy plants and get more yield.

A scientist observed that a particular seed grew very fast after germination. He had recorded growth of plant since germination and he said that its growth can be defined by the function

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 15x + 2, \quad 0 \le x \le 10$$

where x is the number of days the plant is exposed to sunlight.



On the basis of the above information, answer the following questions :

(i) What are the critical points of the function $f(x)$?

(ii) Using second derivative test, find the minimum value of the function.

Sol. **(i)** $f^{I}(x)=x^{2}-8x+15=(x-3)(x-5)$

$f^{I}(x)=0 \Rightarrow x=3,5$ are the critical points.

(ii) Now $f^{II}(x)=2x-8$

$f^{II}(3)<0$ and $f^{II}(5)>0$

so, mimimum value of $f(x)$ is at $x=5$.

$$\text{min. value}=f(5)=\frac{5^{3}}{3}-4(5)^{2}+15(5)+2=\frac{56}{3}$$

prepared by : BALAJI KANCHI

11.

Read the following passage and answer the questions given below:

The relation between the height of the plant ('y' **in cm**) with respect to its exposure to the sunlight is governed by the following equation $y=4x-\frac{1}{2}x^{2}$, where 'x' is the number of days exposed to the sunlight, for $x \le 3$.

(i) Find the rate of growth of the plant with respect to the number of days exposed to the sunlight.

(ii) Does the rate of growth of the plant increase or decrease in the first three days?
What will be the height of the plant after 2 days?

Sol.

$y = 4x - \frac{1}{2}x^2$

(i) The rate of growth of the plant with respect to the number of days exposed to sunlight is given by $\frac{dy}{dx} = 4 - x.$

(ii) Let rate of growth be represented by the function $g(x) = \frac{dy}{dx}$.

Now, $g'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -1 < 0$

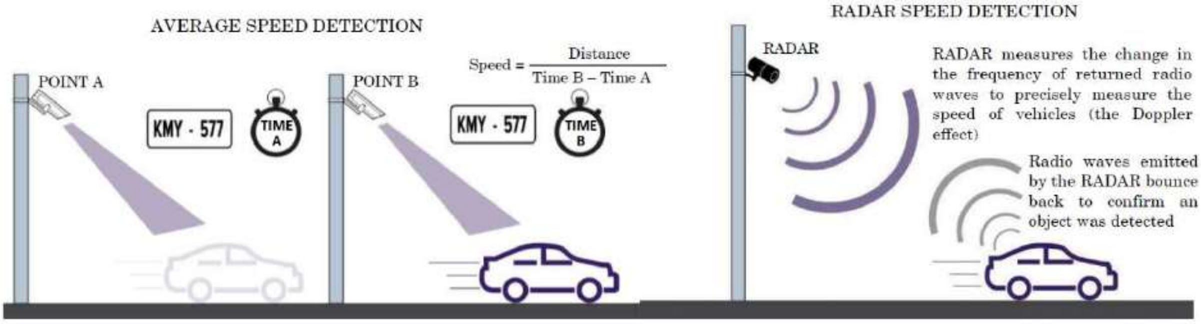
$\Rightarrow g(x)$ decreases.

So the rate of growth of the plant decreases for the first three day

Height of the plant after 2 days is $y = 4 \times 2 - \frac{1}{2}(2)^2 = 6cm.$

12.

The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

On the basis of the above information, answer the following questions :

(i) Express θ in terms of height of the camera installed on the pole and x. *1*

(ii) Find $\frac{d\theta}{dx}$. *1*

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole. *2*

OR

(iii) (b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car. *2*

Sol.

(i) $\tan\theta = \frac{5}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{x}\right)$

(ii) $\frac{d\theta}{dx} = \frac{-5}{5^2 + x^2}$

(iii) (a) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{-5}{5^2 + x^2} \times 20\Big]_{x=50}$

$= \frac{-100}{2525}$ or $\frac{-4}{101}\ rad/s$

OR

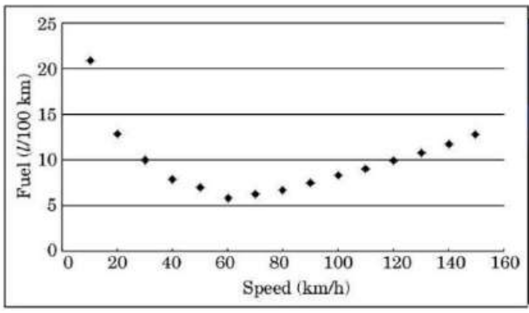
(b) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} \Rightarrow \frac{3}{101} = \frac{-5}{5^2 + x^2}\Big]_{x=50} \times \frac{dx}{dt}$

$\Rightarrow \frac{3}{101} = \frac{-5}{2525} \times \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -15\ m/s$

Hence the speed is 15 m/s

13.

Overspeeding increases fuel consumption and decreases fuel economy as a result of tyre rolling friction and air resistance. While vehicles reach optimal fuel economy at different speeds, fuel mileage usually decreases rapidly at speeds above 80 km/h.



The relation between fuel consumption F (l/100 km) and speed V (km/h) under some constraints is given as $F = \frac{V^2}{500} - \frac{V}{4} + 14$.

On the basis of the above information, answer the following questions :

(i) Find F, when V = 40 km/h. *1*

(ii) Find $\frac{dF}{dV}$. *1*

(iii) (a) Find the speed V for which fuel consumption F is minimum. *2*

OR

(iii) (b) Find the quantity of fuel required to travel 600 km at the speed V at which $\frac{dF}{dV} = -0{\cdot}01$. *2*

Sol.

(i) When V = 40 km/h, F = 36/5 ℓ/100km

(ii)
$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}$$

(iii)(a)
$$\frac{dF}{dV} = 0$$
$\Rightarrow$V = 62.5 km/h

$\frac{d^2F}{dV^2} = \frac{1}{250} > 0$ at V = 62.5 km/h

Hence, F is minimum when V = 62.5 km/h

(iii) (b)
$$\frac{dF}{dV} = -0.01$$
$$\Rightarrow \frac{V}{250} - \frac{1}{4} = \frac{-1}{100}$$
$\Rightarrow V = 60$ km/h

$$F = \frac{60^2}{500} - \frac{60}{4} + 14 = 6.2\,\ell/100km$$

Quantity of fuel required for 600 km
$= 6.2 \times 6 = 37.2\ \ell$

14.

A store has been selling calculators at ₹ 350 each. A market survey indicates that a reduction in price (p) of calculator increases the number of units (x) sold. The relation between the price and quantity sold is given by the demand function $p = 450 - \frac{1}{2}x$.

Based on the above information, answer the following questions :

(i) Determine the number of units (x) that should be sold to maximise the revenue $R(x) = xp(x)$. Also, verify the result.

(ii) What rebate in price of calculator should the store give to maximise the revenue ?

Sol.

(i) Revenue by selling x items $= R(x) = x.p(x) = 450x - \frac{x^2}{2}$

$$\frac{dR}{dx} = 450 - x$$

For Maxima or Minima, $\frac{dR}{dx} = 0 \Rightarrow x = 450$

$$\frac{d^2R}{dx^2} = -1 < 0$$

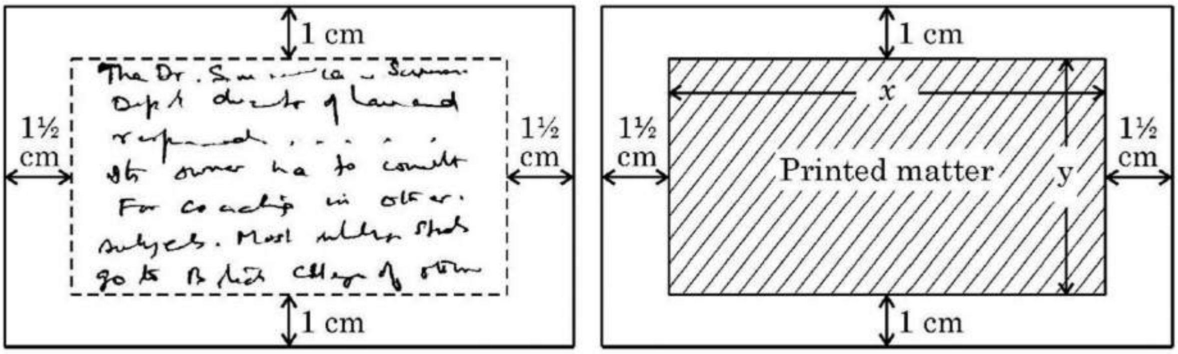
(Revenue is maximum when $x = 450$ units are sold)

(ii) At $x = 450$, $p = 450 - \frac{450}{2} = 225$

So, Rebate $= 350 - 225 =$ Rs.125 per calculator

15. 2024

A rectangular visiting card is to contain 24 sq.cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be 1½ cm as shown below :



On the basis of the above information, answer the following questions :

(i) Write the expression for the area of the visiting card in terms of x.

(ii) Obtain the dimensions of the card of minimum area.

Sol.

(i) Let $A(x)$ be the area of the visiting card then,

$$\text{As } xy = 24,\ A(x) = (x+3)(y+2) = 2x + 3y + xy + 6 = 2x + \frac{72}{x} + 30$$

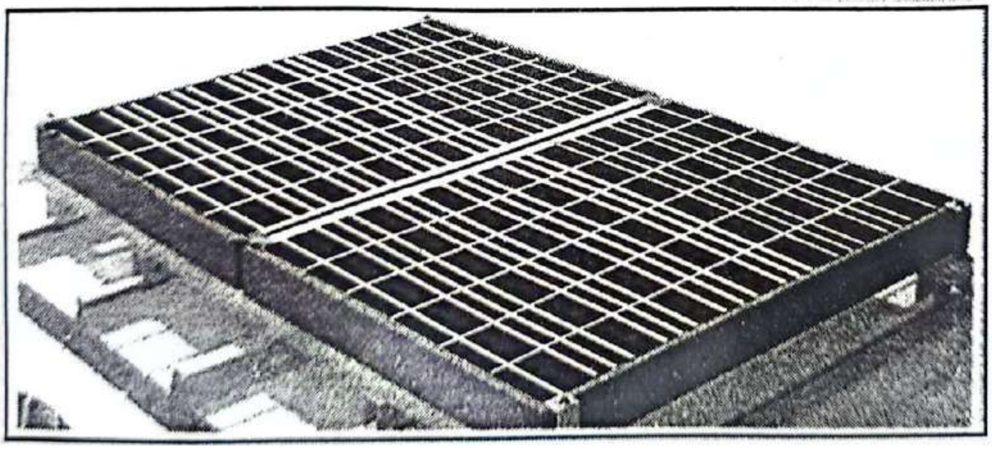
(ii) $A'(x) = 2 - \frac{72}{x^2}$ **and** $A''(x) = \frac{144}{x^3}$,

solving $A'(x) = 0 \Rightarrow x = 6$ **is the critical point.**

$A''(6) = \frac{144}{6} > 0$, $\therefore$ **Area of the card is minimum at** $x = 6, y = 4$

The dimension of the card with minimum area is Length = 9 cm, Breadth = 6 cm

16.



Based on this information, answer the following questions :

(i) Write the equation for the total boundary material used in the boundary and parallel to the partition in terms of x and y.

(ii) Write the area of the solar panel as a function of x.

(iii) (a) Find the critical points of the area function. Use second derivative test to determine critical points at the maximum area. Also, find the maximum area.

OR

(iii) (b) Using first derivative test, calculate the maximum area the company can enclose with the 300 metres of boundary material, considering the parallel partition.

Sol.

$(i)\ 2x+3y=300$

$(ii)\ A=xy=\frac{x}{3}(300-2x)$

$(iii)(a)\ A=\frac{x}{3}(300-2x)=\frac{1}{3}(300x-2x^2)$

$\Rightarrow \frac{dA}{dx}=\frac{1}{3}(300-4x)$

For critical points, put $\frac{dA}{dx}=0 \Rightarrow x=75$

Also, $\frac{d^2A}{dx^2}=-\frac{4}{3}<0$. So, A is maximum at $x=75$

Also, maximum area is $A=\frac{75}{3}(300-150)=3750\text{ m}^2$

$(iii)(b)\ A=\frac{x}{3}(300-2x)=\frac{1}{3}(300x-2x^2)$

$\Rightarrow \frac{dA}{dx}=\frac{1}{3}(300-4x)$

For critical points, put $\frac{dA}{dx}=0 \Rightarrow x=75$

As $\frac{dA}{dx}$ changes its sign from positive to negative as x passes through $x=75$ from left to right, which means $x=75$ is the point of maximum.

Also, maximum area is $A=\frac{75}{3}(300-150)=3750\text{ m}^2$

Note : Full credit to be given if the student takes equation as
$2x+2y=300$ or $2x+4y=300$ or $4x+4y=300$ or $4x+3y=300$
The solutions of sub-parts will differ and marks may be given accordingly.

17.

A small town is analyzing the pattern of a new street light installation. The lights are set up in such a way that the intensity of light at any point x metres from the start of the street can be modelled by $f(x) = e^x \sin x$, where x is in metres.

Based on the above, answer the following :

(i) Find the intervals on which the $f(x)$ is increasing or decreasing, $x \in [0, \pi]$.

(ii) Verify, whether each critical point when $x \in [0, \pi]$ is a point of local maximum or local minimum or a point of inflexion.

Sol.

$f'(x) = e^x(cosx + sinx)$

For critical points, $f'(x) = 0$

$\Rightarrow cosx + sinx = 0$

$\Rightarrow cosx = -sinx$

For x to be a critical point $x \in (0, \pi)$, hence, $x = \frac{3\pi}{4}$

For all $x \in \left[0, \frac{3\pi}{4}\right], f'(x) \geq 0$

Hence, f is increasing in $[0, \frac{3\pi}{4}]$

Note: If a student concludes the answer in any of the following intervals, full marks may be awarded:

$\left(0, \frac{3\pi}{4}\right)$ or $\left[0, \frac{3\pi}{4}\right)$ or $(0, \frac{3\pi}{4}]$

For all $x \in \left[\frac{3\pi}{4}, \pi\right], f'(x) \leq 0$

Hence, f is decreasing in $[\frac{3\pi}{4}, \pi]$

Note: If a student concludes the answer in any of the following intervals, full marks may be awarded:

$(\frac{3\pi}{4}, \pi)$ or $(\frac{3\pi}{4}, \pi]$ or $[\frac{3\pi}{4}, \pi)$

18.

A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.

On the basis of the above information, answer the following questions :

(i) Taking length = breadth = x m and height = y m, express the surface area (S) of the box in terms of x and its volume (V), which is constant.

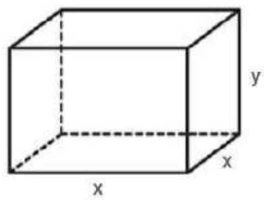
(ii) Find $\frac{dS}{dx}$.

(iii) (a) Find a relation between x and y such that the surface area (S) is minimum.

OR

(iii) (b) If surface area (S) is constant, the volume (V) = $\frac{1}{4}(Sx - 2x^3)$, x being the edge of base. Show that volume (V) is maximum for $x = \sqrt{\frac{S}{6}}$.

Sol.



(i) $V = x^2y \Rightarrow y = \frac{V}{x^2}$ ……………. (i)

Hence, $S = 2x^2 + 4xy = 2x^2 + \frac{4V}{x}$

(ii) $\frac{dS}{dx} = 4\left(x - \frac{V}{x^2}\right)$

(iii) (a) $\frac{dS}{dx} = 0 \Rightarrow V = x^3 \Rightarrow x^2y = x^3 \Rightarrow y = x$

$\frac{d^2S}{dx^2} = 4\left(1 + \frac{2V}{x^3}\right) > 0 \Rightarrow$ S is minimum if $y = x$.

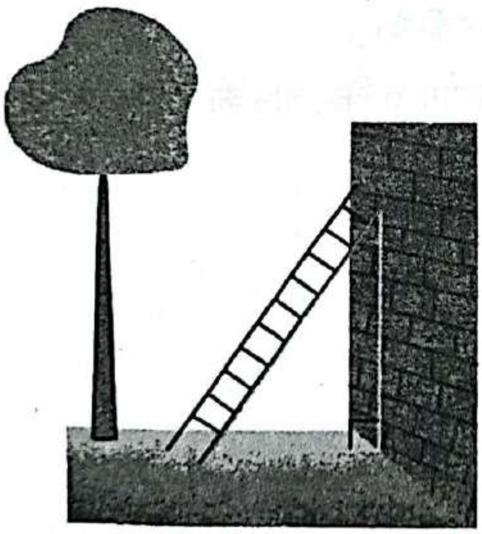
OR

(iii) (b) $V = \frac{1}{4}(Sx - 2x^3) \Rightarrow \frac{dV}{dx} = \frac{1}{4}(S - 6x^2)$

Put $\frac{dV}{dx} = 0 \Rightarrow x = \sqrt{\frac{S}{6}}$

$\left(\frac{d^2V}{dx^2}\right)_{x=\sqrt{\frac{S}{6}}} = -3\sqrt{\frac{S}{6}} < 0 \Rightarrow$ Volume is maximum for $x = \sqrt{\frac{S}{6}}$.

19.



A ladder of fixed length 'h' is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions :

(i) Express the distance (y) between the wall and foot of the ladder in terms of 'h' and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer.

(ii) Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point.

(iii) (a) Show that the area (A) of the right triangle is maximum at the critical point.

OR

(iii) (b) If the foot of the ladder whose length is 5 m, is being pulled towards the wall such that the rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the ladder is 3 m away from the wall ?

Sol.

$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$

ii) $\frac{dA}{dx} = \frac{1}{2}\sqrt{h^2 - x^2} + \frac{1}{2}x\frac{-x}{\sqrt{h^2 - x^2}}$

$\frac{dA}{dx} = 0 \text{ gives } x = \frac{h}{\sqrt{2}}$

(iii)(a) $A'' = \frac{1}{2}\frac{-4x.\sqrt{h^2 - x^2} - (h^2 - 2x^2)\frac{-x}{\sqrt{h^2 - x^2}}}{h^2 - x^2}$ is < 0 at $x = \frac{h}{\sqrt{2}}$

Hence A is maximum at critical point

OR

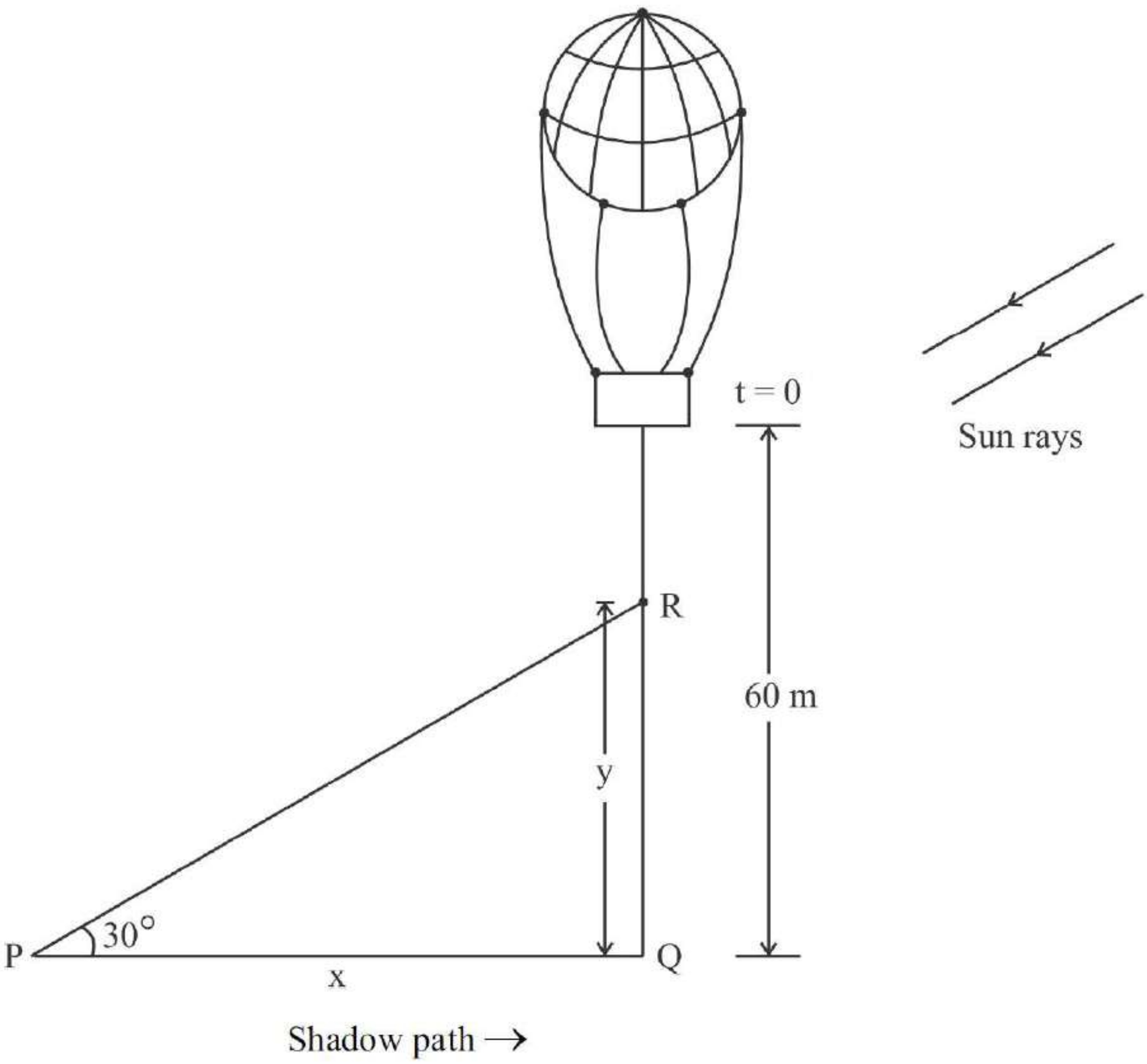
(iii)(b) $y^2 = 25 - x^2$ hence $y = 3$ gives $x = 4$

$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$\frac{dx}{dt} = 1.5 \text{m/s}$

20.

A sandbag is dropped from a balloon at a height of 60 metres.



When the angle of elevation of the sun is 30°, the position of the sandbag is given by the equation $y = 60 - 4.9\,t^2$, where y is the height of the sandbag above the ground and t is the time in seconds.

On the basis of the above information, answer the following questions :

(i) Find the relation between x and y, where x is the distance of the shadow at P from the point Q and y is the height of the sandbag above the ground.

(ii) After how much time will the sandbag be 35 metres above the ground ?

(iii) (a) Find the rate at which the shadow of the sandbag is travelling along the ground when the sandbag is at a height of 35 metres.

OR

(iii) (b) How fast is the height of the sandbag decreasing when 2 seconds have elapsed ?

Sol.

(i) $\frac{y}{x} = \tan 30° \Rightarrow y = \frac{x}{\sqrt{3}}$ or $x = \sqrt{3}y$

(ii) Using $y = 35m \Rightarrow 60 - 4.9t^2 = 35 \Rightarrow 4.9t^2 = 25 \Rightarrow t = \frac{5\sqrt{10}}{7}$ **seconds.**

(iii) (a) $x = \sqrt{3}y \Rightarrow x = 60\sqrt{3} - 4.9\sqrt{3}t^2 \Rightarrow \left.\frac{dx}{dt}\right]_{t=\frac{5\sqrt{10}}{7}} = -4.9\sqrt{3}(2t)\Big]_{t=\frac{5\sqrt{10}}{7}} = -7\sqrt{30}$ **m/s**

(iii) (b) $\frac{dy}{dt} = -9.8t = -9.8 \times 2 = -19.6$ **m/s**

Height of the sandbag is decreasing at the rate of 19.6 m/s

21.

An architect designs a building for a Company. The design of window on the ground floor is proposed to be different than at the other floors. The window is in the shape of a rectangle whose top length is surmounted by a semi-circular opening. This window has a perimeter of 10 m.

Based on the above information, answer the following :

(i) If $2x$ and 2y represent the length and breadth of the rectangular portion of the window, then establish a relation between x and y.

(ii) Find the total area of the window in terms of x.

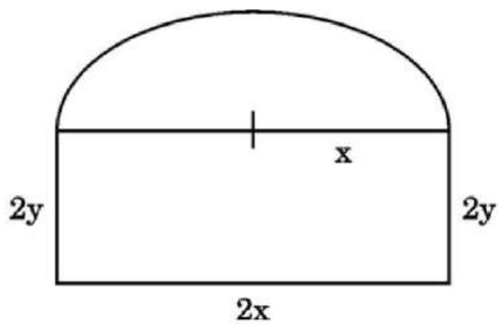
(iii) (a) Find the values of x and y for the maximum area of the window.

OR

(iii) (b) If x and y represent the length and breadth of the rectangle, then establish the expression for the area of the window in terms of x only.

Sol.

(i) $2x + 4y + \pi x = 10$



(ii)

$$A = (2x)(2y) + \frac{1}{2}\pi x^2$$

$$y = \frac{10 - 2x - \pi x}{4}$$

$$A = 4x\frac{(10 - 2x - \pi x)}{4} + \frac{1}{2}\pi x^2$$

$$A = 10x - 2x^2 - \frac{1}{2}\pi x^2$$

(iii) (a) $\frac{dA}{dx} = 10 - 4x - \pi x = 0$

$\Rightarrow 10 = (\pi + 4)x$

$$\Rightarrow x = \frac{10}{\pi + 4}$$

$\frac{d^2y}{dx^2} =$ -4 - $\pi < 0$

$$\Rightarrow y = \frac{1}{4}\left[10 - 2\left(\frac{10}{\pi + 4}\right) - \pi\left(\frac{10}{\pi + 4}\right)\right]$$

$$\Rightarrow y = \frac{1}{4}\frac{[10\pi + 40 - 20 - 10\pi]}{\pi + 4}$$

$$y = \frac{1}{4}\left[\frac{20}{\pi + 4}\right] = \frac{5}{\pi + 4}$$

(iii) (b) $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$$x + 2y + \pi\left(\frac{x}{2}\right) = 10$$

$$\Rightarrow y = \frac{20 - 2x - \pi x}{4}$$

$$\therefore A = x\left(\frac{20 - 2x - \pi x}{4}\right) + \frac{1}{8}\pi x^2$$

$$A = 5x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$$

22.

A window is in the form of a rectangle surmounted by an equilateral triangle on its length. Let the rectangular part have length and breadth x and y metres respectively.

Based on the given information, answer the following questions :

(i) If the perimeter of the window is 12 m, find the relation between x and y.

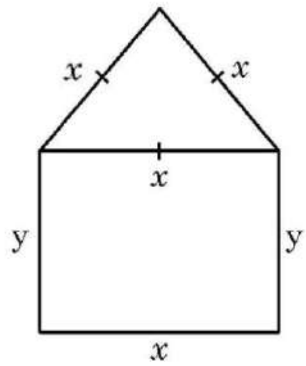
(ii) Using the expression obtained in (i), write an expression for the area of the window as a function of x only.

(iii) (a) Find the dimensions of the rectangle that will allow maximum light through the window. (use expression obtained in (ii))

OR

(iii) (b) If it is given that the area of the window is 50 m^2, find an expression for its perimeter in terms of x.

$(i)\,\text{Perimeter}(P) = 3x + 2y = 12$



$$(ii)\,\text{Area}(A) = xy + \frac{\sqrt{3}}{4}x^2$$

$$= x\left(\frac{12-3x}{2}\right) + \frac{\sqrt{3}}{4}x^2$$

$$= 6x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2$$

$$(iii)(a)\,\frac{dA}{dx} = 6 - 3x + \frac{\sqrt{3}}{2}x$$

For maximum light, $\frac{dA}{dx}=0$

$\Rightarrow 6-3x+\frac{\sqrt{3}}{2}x=0 \Rightarrow x=\frac{12}{6-\sqrt{3}}m$

Also, $\frac{d^2A}{dx^2}=-3+\frac{\sqrt{3}}{2}<0 \therefore A$ is maximum when $x=\frac{12}{6-\sqrt{3}}m$

Now, $y=\frac{12-3x}{2}=6-\frac{3}{2}\left(\frac{12}{6-\sqrt{3}}\right)=\frac{18-6\sqrt{3}}{6-\sqrt{3}}m$

OR

$(iii)(b)\, xy+\frac{\sqrt{3}}{4}x^2=50$

$\Rightarrow y=\frac{50}{x}-\frac{\sqrt{3}}{4}x$

Now, $P=3x+2y$

$$=3x+2\left(\frac{50}{x}-\frac{\sqrt{3}}{4}x\right)m$$

23.

A fighter-jet of the enemy is flying along the parabolic path $4y = x^2$. A soldier is located at the point (0, 5) and is aiming to shoot down the jet when it is nearest to him.

Based on the above, answer the following questions :

(i) Let (x, y) be the position of the jet at any instant. Express the distance between the soldier and the jet as the function f(x).

(ii) Taking $S = [f(x)]^2$, find $\frac{dS}{dx}$.

(iii) (a) What will be the position of the jet when the soldier shoots it down ?

OR

(iii) (b) What will be the distance between the soldier and the jet at the instant when he shoots it down ?

Sol.

$(i)\ \text{Distance} = \sqrt{(x-0)^2 + \left(\frac{x^2}{4} - 5\right)^2} = f(x)$

$(ii)\ S = [f(x)]^2 = x^2 + \left(\frac{x^2}{4} - 5\right)^2$

$\Rightarrow \frac{dS}{dx} = 2x + 2\left(\frac{x^2}{4} - 5\right)\left(\frac{x}{2}\right) = \frac{1}{4}x\left(x^2 - 12\right)$

$(iii)(a)\ \frac{dS}{dx} = 0 \Rightarrow x = 0, \pm 2\sqrt{3}$

Showing $\left(2\sqrt{3}, 3\right)$ and $\left(-2\sqrt{3}, 3\right)$ are the point of minima.

$\therefore$ Required position is $\left(\pm 2\sqrt{3}, 3\right)$.

OR

(b) getting $x = \pm 2\sqrt{3}$ as a point of minima,

Distance $= \sqrt{\left(\pm 2\sqrt{3}\right)^2 + (3-5)^2} = 4$ units