@kanchibalaji7
+91-8099454846
@ikbmaths7

7.1 Indefinite Integration :

(Class XII CBSE Board Exam Models from 2022-2025 with solutions)

2 mark :

standard integral formulas :

1.

Find :

$$\int \sqrt{3 - 2x - x^2}\, dx$$

Sol.

$$\int \sqrt{3 - 2x - x^2}\,dx = \int \sqrt{2^2 - (x+1)^2}\,dx$$

$$= \frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\frac{x+1}{2} + C$$

I. Substitution method :

a. Algebraic substitution :

1. 2022

Find : $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

Sol. $\displaystyle I = \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$

$$= \int \frac{6y^5\,dy}{y^3 + y^2} \qquad x = y^6 \text{ so that } dx = 6y^5 dy$$

$$= 6\int \frac{y^3}{y+1}\,dy$$

$$= 6\int \left[(y^2 - y + 1) - \frac{1}{y+1}\right] dy$$

$$= 6\left[\frac{y^3}{3} - \frac{y^2}{2} + y - \log|y+1|\right] + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6x^{1/6} - 6\log(x^{1/6}+1) + C$$

2.

Evaluate :

$$\int_{1/3}^{1} \frac{(x - x^3)^{1/3}}{x^4}\, dx$$

Sol.

$$I = \int_{1/3}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4}\, dx$$

$$= \int_{1/3}^{1} \frac{(x^3)^{1/3}\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4}\, dx = \int_{1/3}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3}\, dx$$

Put $\left(\frac{1}{x^2} - 1\right) = t$ so that $\frac{-2}{x^3}\,dx = dt$

Thus,

$$I = \int_0^8 t^{\frac{1}{3}} \times \frac{dt}{(2)}$$

$$= \frac{48}{8} = 6.$$

prepared by : BALAJI KANCHI

@kanchibalaji7
+91-8099454846
@ikbmaths7

3.

Find :

$$\int x\sqrt{1+2x}\ dx$$

Sol.

$1 + 2x = t^2$

$2\,dx = 2t\,dt$

$\frac{1}{2}\int(t^4 - t^2)dt = \frac{1}{2}\left[\frac{t^5}{5} - \frac{t^3}{3}\right] + C$

$$= \frac{(1+2x)^{\frac{5}{2}}}{10} - \frac{(1+2x)^{\frac{3}{2}}}{6} + C$$

6. 2025

$$\int \frac{\sqrt{x}}{1+\sqrt{x^{3/2}}}dx$$

Sol.

$$I=\int \frac{\sqrt{x}}{1+\sqrt{x^{3/2}}}dx$$

$$\text{Put } x^{3/2}=t \Rightarrow \frac{3}{2}\sqrt{x}dx=dt$$

$$\Rightarrow I=\frac{2}{3}\int \frac{dt}{1+\sqrt{t}}$$

$$\text{Put } \sqrt{t}=z \Rightarrow \frac{1}{2\sqrt{t}}dt=dz$$

$$\Rightarrow I=\frac{2}{3}\int \frac{2z}{1+z}dz$$

$$=\frac{4}{3}\left[\int 1dz-\int \frac{1}{1+z}dz\right]$$

$$=\frac{4}{3}\left[z-\log|1+z|\right]+C$$

$$=\frac{4}{3}\left[\sqrt{t}-\log\left|1+\sqrt{t}\right|\right]+C$$

$$=\frac{4}{3}\left[\sqrt{x^{3/2}}-\log\left|1+\sqrt{x^{3/2}}\right|\right]+C$$

b. Based on Trigonometric Substitution :

1.

Find : $\int \cos^3 x\, e^{\log \sin x}\, dx$

Sol.

$$\int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \cdot \sin x\, dx \text{, Assuming } \cos x = t \text{ and } \sin x\, dx = -dt$$
$$= -\int t^3 dt$$
$$= -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

2.

Find :

$$\int \sin^3 x \,.\, \cos^4 x\, dx$$

Sol.

$$\int \sin^3 x \cos^4 x\, dx = \int (1 - \cos^2 x) \cos^4 x \sin x\, dx$$
$$= -\int (1 - t^2) t^4\, dt \qquad (\text{putting } \cos x = t)$$
$$= -\int (t^4 - t^6)\, dt$$
$$= \int (t^6 - t^4)\, dt$$
$$= \frac{t^7}{7} - \frac{t^5}{5} + C$$
$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

3.

$$\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx$$

Sol. Let $\cot x = t$, then $-\operatorname{cosec}^2 x dx = dt$

$$\therefore \int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = \int \frac{\sqrt{\cot x}}{\cot x} \operatorname{cosec}^2 x \, dx = -\int \frac{\sqrt{t}}{t} dt = -\int \frac{1}{\sqrt{t}} dt$$

$$= -2\sqrt{t} + C$$

$$= -2\sqrt{\cot x} + C$$

4.

Evaluate :

$$\int \left(\sqrt{\cot x} + \sqrt{\tan x}\right) dx$$

$$I = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

On squaring both sides, we get $1 - \sin 2x = t^2$

$$I = \sqrt{2} \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= \sqrt{2} \sin^{-1} t + C$$

$$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C$$

5.

Find : $\int \frac{dx}{\sin x + \sin 2x}$

Sol.

$$I = \int \frac{dx}{\sin x + \sin 2x}$$

$$= \int \frac{dx}{\sin x(1 + 2\cos x)}$$

$$= \int \frac{\sin x}{\sin^2 x(1 + 2\cos x)} dx$$

$$= \int \frac{\sin x}{(1 - \cos x)(1 + \cos x)(1 + 2\cos x)} dx$$

Put $\cos x = t \Rightarrow \sin x\, dx = dt$

$$I = -\int \frac{dt}{(1 - t)(1 + t)(1 + 2t)}$$

$$= -\frac{1}{6}\int \frac{dt}{1 - t} + \frac{1}{2}\int \frac{dt}{1 + t} - \frac{4}{3}\int \frac{dt}{1 + 2t}$$

$$= \frac{1}{6}\log|1 - t| + \frac{1}{2}\log|1 + t| - \frac{2}{3}\log|1 + 2t| + C$$

$$= \frac{1}{6}\log|1 - \cos x| + \frac{1}{2}\log|1 + \cos x| - \frac{2}{3}\log|1 + 2\cos x| + C$$

6.

If $\frac{d}{dx}[F(x)] = \frac{\sec^4 x}{\operatorname{cosec}^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find F(x).

Sol. $F(x) = \int \tan^4 x\, dx = \int \tan^2 x.(\sec^2 x - 1)\, dx$

$$= \int (\tan^2 x \sec^2 x - \sec^2 x + 1)\, dx$$

$$F(x) = \frac{\tan^3 x}{3} - \tan x + x + C$$

$$x = \frac{\pi}{4},\ F(x) = \frac{\pi}{4} \text{ gives } C = \frac{2}{3}$$

$$F(x) = \frac{\tan^3 x}{3} - \tan x + x + \frac{2}{3}$$

c. Based on Trigonometric formulas :

1.

$$\int \sin 2x \sin 3x\, dx$$

Ans. $I = \int \sin 2x \sin 3x\, dx$

$$= \frac{1}{2}\int 2 \sin 2x \sin 3x\, dx$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$$

2.

Find :

$$\int \frac{1}{\cos(x-a)\ \cos(x-b)}\,dx$$

Sol.

$$\text{Let } I = \int \frac{1}{\cos(x-a)\ \cos(x-b)}\,dx$$

$$= \frac{1}{sin(a-b)} \int \frac{sin[(x-b)-(x-a)]}{cos(x-a)\ \cos(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \left[\int \frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\ \cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\ \cos(x-b)}\right]dx$$

$$= \frac{1}{\sin(a-b)} \left[\int [\tan(x-b) - \tan(x-a)]\,dx\right]$$

$$= \frac{1}{\sin(a-b)} [\log|\sec(x-b)| - \log|\sec(x-a)|] + C$$

3.

$$\int \frac{\sin 3x}{\sin x}\,dx$$

Sol.

$$\int \frac{\sin 3x}{\sin x}\,dx = \int \frac{3\sin x - 4\sin^3 x}{\sin x}\,dx$$

$$= \int [3 - 4\frac{(1-\cos 2x)}{2}]dx$$

$$= \int (1 + 2\cos 2x)\,dx$$

$$= x + \sin 2x + C$$

4.

Find :

$$\int \frac{\cos x}{\sin 3x} dx$$

Sol.

$$I = \int \frac{\cos x}{3 \sin x - 4 \sin^3 x} dx$$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$I = \int \frac{dt}{3t - 4t^3}$$

$$= \int \frac{1}{t^3\left(\frac{3}{t^2} - 4\right)} dt$$

Let $\frac{3}{t^2} - 4 = z \Rightarrow -\frac{6}{t^3} dt = dz$

$$I = -\frac{1}{6} \int \frac{dz}{z}$$

$$= -\frac{1}{6} \log |z| + C$$

$$= -\frac{1}{6} \log |3 \operatorname{cosec}^2 x - 4| + C$$

prepared by : BALAJI KANCHI

@kanchibalaji7
+91-8099454846
@ikbmaths7

5.

$$\int \frac{\sin x}{\sin(x-2a)}\,dx$$

Ans. $I = \int \frac{\sin x}{\sin(x-2a)}\,dx$

$= \int \frac{\sin[(x-2a)+2a]}{\sin(x-2a)}\,dx$

$= \int \frac{\sin(x-2a)\cos 2a + \cos(x-2a)\sin 2a}{\sin(x-2a)}\,dx$

$= \int \cos 2a\,dx + \sin 2a \int \cot(x-2a)\,dx$

$= x\cos 2a + \sin 2a \cdot \log|\sin(x-2a)| + C$

6.

Evaluate : $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\;dx$

Sol.

Given definite integral $= \int_0^{\frac{\pi}{4}} \sqrt{(sinx + cosx)^2}dx$

$= \int_0^{\frac{\pi}{4}} (sinx + cosx)dx$

$= [-cosx + sinx]_0^{\frac{\pi}{4}}$

$= 1$

7.

Find : $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\, dx$

Sol.

$$\int \frac{\cos 2x}{(\sin x + \cos x)^2}dx = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}dx$$

$$= \int \frac{\cos x - \sin x}{\sin x + \cos x}dx$$

$$= \log|\sin x + \cos x| + C$$

8.

Find :

$$\int \frac{\cos x\, dx}{1 + \cos x + \sin x}$$

Sol.

$$I = \int \frac{\cos x}{1 + \cos x + \sin x}dx$$

$$= \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2}\cos \frac{x}{2}}dx$$

$$= \int \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{2\cos \frac{x}{2}}dx$$

$$= \frac{1}{2}\int \left(1 - \tan \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left[x + 2\log\cos \frac{x}{2}\right] + C$$

9.

Find : $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\ dx.$

Sol.

Put $x = a\tan^2\theta,\ dx = 2a\tan\theta\sec^2\theta\ d\theta$

$$\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx = 2a\int \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}\left(\tan\theta\cdot\sec^2\theta\right)d\theta$$

$$= 2a\int \theta\cdot\left(\tan\theta\sec^2\theta\right)d\theta$$

$$= 2a\left[\theta\frac{(\tan\theta)^2}{2} - \frac{1}{2}\int(\tan\theta)^2 d\theta\right]$$

$$= a\left[\theta\tan^2\theta - (\tan\theta - \theta)\right] + C$$

$$= a\left[\tan^{-1}\left(\sqrt{\frac{x}{a}}\right)\cdot\frac{x}{a} - \sqrt{\frac{x}{a}} + \tan^{-1}\left(\sqrt{\frac{x}{a}}\right)\right] + C$$

$$\text{or } (a+x)\tan^{-1}\left(\sqrt{\frac{x}{a}}\right) - \sqrt{ax} + C$$

d. $\int \frac{f'(x)}{f(x)}\, dx$ *model* :

1.

.

Find :

$$\int \frac{dx}{1+\cot x}$$

Sol.

$$I=\int \frac{dx}{1+\cot x}=\int \frac{\sin x}{\sin x+\cos x}\,dx$$

$$=\frac{1}{2}\int \frac{(\sin x+\cos x)-(\cos x-\sin x)}{\sin x+\cos x}\,dx$$

$$=\frac{1}{2}\int 1.dx+\frac{1}{2}\int \frac{\sin x-\cos x}{\sin x+\cos x}\,dx$$

$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

2.

Find :

$$\int \frac{e^{4x}-1}{e^{4x}+1}\,dx$$

Sol.

Given integral $=\int \frac{e^{4x}-1}{e^{4x}+1}\,dx$

$$=\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$$

$$=\frac{1}{2}\int \frac{2(e^{2x}-e^{-2x})}{e^{2x}+e^{-2x}}\,dx$$

$$=\frac{1}{2}log|e^{2x}+e^{-2x}|+C$$

\

3.

$$\int \frac{1}{e^x + 1}\, dx$$

Sol.

Let, $e^x = t$, $e^x dx = dt$

$$\int \frac{1}{e^x+1} dx = \int \frac{1}{t(t+1)} dt$$

$$= \int \frac{dt}{t} - \int \frac{dt}{t+1}$$

$$= \log|t| - \log|t+1| + C$$

$$= \log e^x - \log(e^x+1) + C \text{ or } \log\left|\frac{e^x}{1+e^x}\right| + C$$

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e. $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx$ *model* :

1.

Find $\int \frac{dx}{\sqrt{\sin^3 x \cos(x-\alpha)}}$

Sol.

$$\text{Let } I=\int \frac{1}{\sqrt{\sin^3 x\cos(x-\alpha)}}dx=\int \frac{\operatorname{cosec}^2 x}{\sqrt{\sin\alpha+\cos\alpha\cot x}}dx$$

$$\text{Put } \sin\alpha+\cos\alpha\cot x=t \Rightarrow \operatorname{cosec}^2 x\,dx=-\frac{1}{\cos\alpha}dt$$

$$\therefore I=-\int \frac{1}{\cos\alpha\sqrt{t}}dt=-\frac{2\sqrt{t}}{\cos\alpha}+c$$

$$\Rightarrow I=-\frac{2}{\cos\alpha}\sqrt{\sin\alpha+\cos\alpha\cot x}+c$$

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II. Formula forms :

$$\int \frac{1}{x^2-a^2}dx\ ,\ \int \frac{1}{x^2+a^2}dx,\ \int \frac{1}{a^2-x^2}dx,\ \int \frac{1}{\sqrt{x^2-a^2}}dx\ ,\ \int \frac{1}{\sqrt{x^2+a^2}}dx\ ,\ \int \frac{1}{\sqrt{a^2-x^2}}dx\ :$$

a.Direct form :

1.

Find : $\int \frac{1}{x}\sqrt{\frac{x+a}{x-a}}\,dx.$

Sol.

$$I = \int \frac{1}{x}\frac{x+a}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{x^2-a^2}}dx + a\int \frac{1}{x\sqrt{x^2-a^2}}dx$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| + \sec^{-1}\left(\frac{x}{a}\right) + C$$

b. After substitution

$\int \frac{1}{x^2-a^2}dx$, $\int \frac{1}{x^2+a^2}dx$, $\int \frac{1}{a^2-x^2}dx$, $\int \frac{1}{\sqrt{x^2-a^2}}dx$, $\int \frac{1}{\sqrt{x^2+a^2}}dx$, $\int \frac{1}{\sqrt{a^2-x^2}}dx$ form:

1.

Find $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$.

Sol.

$$\int \frac{\sin^{-1}x}{\left(1-x^2\right)^{3/2}}dx = \int t\sec^2 t\,dt\text{ , (Putting }\sin^{-1}x = t, x = \sin t\text{, also }\frac{1}{\sqrt{1-x^2}}dx = dt\text{)}$$

$$= t\cdot\tan t + \log|\cos t| + c$$

$$= \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log\left|1-x^2\right| + c$$

2.

Find :

$$\int \frac{dx}{\cos x\sqrt{\cos 2x}}$$

Sol.

$$\int \frac{dx}{\cos x\sqrt{\cos 2x}} = \int \frac{dx}{\cos x\sqrt{\cos^2 x - \sin^2 x}}$$

$$= \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$$

$$= \int \frac{1}{\sqrt{1-t^2}}dt\text{ , using }\tan x = t, \sec^2 x\cdot dx = dt$$

$$= \sin^{-1}t + c = \sin^{-1}\left(\tan x\right) + c$$

@kanchibalaji7
+91-8099454846
@ikbmaths7

III. Quadratic Polynomial :

$\int \frac{1}{ax^2+bx+c}dx$ / $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$ / $\int \frac{px+q}{ax^2+bx+c}dx$ / $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$:

a . $\int \frac{1}{ax^2+bx+c}dx$ form :

1.

Find : $\int \frac{dx}{x^2-6x+13}$

Sol.

$$\int \frac{dx}{x^2-6x+13}$$

$$= \int \frac{dx}{(x-3)^2+(2)^2}$$

$$= \frac{1}{2}tan^{-1}\frac{(x-3)}{2} + C$$

2.

Find : $\int \frac{1}{5+4x-x^2}dx$

Sol.

$$\int \frac{1}{5+4x-x^2}dx = \int \frac{1}{3^2-(x-2)^2}dx$$

$$= \frac{1}{6}\log\left|\frac{1+x}{5-x}\right| + C$$

@kanchibalaji7
+91-8099454846
@ikbmaths7

b. $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$ form :

1.

Find : $\int \frac{1}{\sqrt{5-6x-x^2}}dx$.

$$\text{Given integral} = \int \left(\frac{dx}{\sqrt{14-(x+3)^2}}\right)$$
$$= \sin^{-1}\frac{x+3}{\sqrt{14}} + C$$

2.

$$\int \frac{dx}{\sqrt{4x-x^2}}$$
$$= \int \frac{dx}{\sqrt{2^2-(x-2)^2}}$$
$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

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2.c

Given $\frac{d}{dx}F(x) = \frac{1}{\sqrt{2x-x^2}}$ and $F(1) = 0$, find $F(x)$.

Sol.

$$F(x) = \int \frac{1}{\sqrt{2x-x^2}}dx$$
$$= \int \frac{1}{\sqrt{1-(x-1)^2}}dx$$
$$= \sin^{-1}(x-1) + c$$

when $x=1, y=0$ gives $c=0$

$$\therefore F(x) = \sin^{-1}(x-1)$$

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c. $\int \frac{px+q}{ax^2+bx+c}dx$ *form* :

1.

$$\int \frac{2x}{x^2 + 3x + 2}\,dx$$

Sol.

$$I = \int \frac{2x \cdot dx}{x^2 + 3x + 2} = \int \frac{2x}{(x+1)(x+2)}dx$$

$$= \int \left(\frac{-2}{x+1} + \frac{4}{x+2}\right)dx \qquad \text{using partial fraction}$$

$$= -2\log|x+1| + 4\log|x+2| + C$$

d. $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ *form* :

1.

Find : $\displaystyle\int \frac{3x+5}{\sqrt{x^2+2x+4}}\,dx$

Sol.

$$I = \int \frac{3x+5}{\sqrt{x^2+2x+4}}dx = \frac{3}{2}\int \frac{2x+2}{\sqrt{x^2+2x+4}}dx + 2\int \frac{1}{\sqrt{x^2+2x+4}}dx$$

$$I = \frac{3}{2}(2\sqrt{x^2+2x+4}) + 2\int \frac{1}{\sqrt{(x+1)^2+\sqrt{3}^2}}dx$$

$$I = 3\sqrt{x^2+2x+4} + 2\log|(x+1) + \sqrt{x^2+2x+4}| + c$$

prepared by : BALAJI KANCHI

@kanchibalaji7
+91-8099454846
@ikbmaths7

2.

Find $\int \frac{x+2}{\sqrt{x^2-4x-5}}\,dx.$

Sol.

$$\int \frac{x+2}{\sqrt{x^2-4x-5}}dx = \frac{1}{2}\int \frac{2x-4}{\sqrt{x^2-4x-5}}dx + 4\int \frac{1}{\sqrt{(x-2)^2-3^2}}dx$$

$$= \sqrt{x^2-4x-5} + 4\log\left|x-2+\sqrt{x^2-4x-5}\right| + c$$

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3.

Find :

$$\int \frac{5x-3}{\sqrt{1+4x-2x^2}}\,dx$$

Sol.

$$\int \frac{5x-3}{\sqrt{1+4x-2x^2}}dx = -\frac{5}{4}\int \frac{-4x+4}{\sqrt{1+4x-2x^2}}dx + \sqrt{2}\int \frac{1}{\sqrt{\left(\left(\sqrt{\frac{3}{2}}\right)^2-(x-1)^2\right)}}dx$$

$$= -\frac{5}{2}\sqrt{1+4x-2x^2} + \sqrt{2}\sin^{-1}\left(\sqrt{\frac{2}{3}}(x-1)\right) + c$$

e. After substitution

$\int \frac{1}{ax^2+bx+c}dx \;/\; \int \frac{1}{\sqrt{ax^2+bx+c}}dx \;/\; \int \frac{px+q}{ax^2+bx+c}dx \;/\; \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx \;\; form:$

Trgonometric substitution :

1.

Find :

$$\int \frac{\cos\theta}{\sqrt{3-3\sin\theta-\cos^2\theta}}\,d\theta$$

Sol.

$$I=\int\frac{\cos\theta}{\sqrt{3-3\sin\theta-\cos^2\theta}}d\theta$$

$$=\int\frac{\cos\theta}{\sqrt{\sin^2\theta-3\sin\theta+2}}d\theta$$

Put $\sin\theta=t \Rightarrow \cos\theta\,d\theta=dt$

$$I=\int\frac{dt}{\sqrt{t^2-3t+2}}=\int\frac{dt}{\sqrt{\left(t-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=\log\left|\left(t-\frac{3}{2}\right)+\sqrt{t^2-3t+2}\right|+C$$

$$=\log\left|\left(\sin\theta-\frac{3}{2}\right)+\sqrt{\sin^2\theta-3\sin\theta+2}\right|+C$$

Exponential substitution :

1.

Find :

$$\int \frac{e^{x}}{\sqrt{e^{2x} - 4e^{x} - 5}}\,dx$$

Sol.

Let $e^{x} = t$. Then $e^{x}\,dx = dt$

Given integral becomes

$$\int \frac{dt}{\sqrt{t^2 - 4t - 5}}$$

$$= \int \frac{dt}{\sqrt{(t-2)^2 - 3^2}}$$

$$= \log\left|(t-2) + \sqrt{t^2 - 4t - 5}\right| + C$$

$$= \log\left|e^{x} - 2 + \sqrt{e^{2x} - 4e^{x} - 5}\right| + C$$

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2.

Integrate :

$$\frac{e^x}{\sqrt{5-4e^x-e^{2x}}} \text{ with respect to x.}$$

Sol.

Let $e^x = t$, so that $e^x\,dx = dt$. Then,

$$I = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{3^2-(t+2)^2}}$$

$$= \sin^{-1}\left(\frac{t+2}{3}\right) + C$$

$$= \sin^{-1}\left(\frac{e^x+2}{3}\right) + C$$

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logarithmic substitution :

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Find :

$$\int \frac{1}{x[(\log x)^2 - 3\log x - 4]}\,dx$$

Sol.

Let $logx = t \Rightarrow \frac{1}{x}dx = dt$

The given integral becomes $= \int \frac{1}{t^2-3t-4}dt$

$$= \int \frac{1}{\left(t-\frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}dx$$

$$= \frac{1}{5}log\left|\frac{t-4}{t+1}\right| + C$$

$$= \frac{1}{5}log\left|\frac{logx-4}{logx+1}\right| + C$$

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Find :

$$\int \frac{x^{-1}}{(\log x)^2 - 5\log x + 4}\, dx$$

Sol.

Let $\log x = t;\ \frac{1}{x}\,dx = dt$

$$\text{Given integral} = \int \frac{1}{t^2 - 5t + 4}\, dt$$

$$= \int \frac{1}{\left(t - \frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\, dt$$

$$= \frac{1}{3} \log \left| \frac{t-4}{t-1} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{\log x - 4}{\log x - 1} \right| + C$$

IV. Partial fractions :

In the denominator :-

a. Linear factors in the denominator :

1.

Find : $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\, dx$

Sol.

$$\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx = -\frac{1}{6}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{2}\int \frac{1}{x-3}dx$$

(Using Partial Fraction)

$$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$$

b. Repeated linear factors :

1.

Find : $\int \frac{2x+1}{(x+1)^2(x-1)}dx$

Sol.

$$\int \frac{2x+1}{(x+1)^2(x-1)}dx = -\frac{3}{4}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}dx + \frac{3}{4}\int \frac{1}{x-1}dx$$

$$= -\frac{3}{4}\log|x+1| - \frac{1}{2(x+1)} + \frac{3}{4}\log|x-1| + C$$

$$= \frac{3}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2(x+1)} + C$$

2.

Find :

$$\int \frac{3x+1}{(x-2)^2\,(x+2)}\,dx$$

Sol.

$$\int \frac{3x+1}{(x-2)^2(x+2)}dx = \frac{5}{16}\int \frac{1}{x-2}dx + \frac{7}{4}\int \frac{1}{(x-2)^2}dx - \frac{5}{16}\int \frac{1}{x+2}dx$$

$$= \frac{5}{16}\log|x-2| - \frac{7}{4(x-2)} - \frac{5}{16}\log|x+2| + C$$

$$\text{or } = \frac{5}{16}\log\left|\frac{x-2}{x+2}\right| - \frac{7}{4(x-2)} + C$$

2.

Find :

$$\int \frac{x^2+1}{(x-1)^2\,(x+3)}\,dx$$

Sol.

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} = \frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} + \frac{5/8}{x+3}$$

$$I = \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$$

3.

Find :

$$\int \frac{x^2 + x + 1}{(x+1)^2 (x+2)} dx$$

Sol.

$$\text{Let } I = \int \frac{x^2 + x + 1}{(x + 1)^2 \ (x + 2)} \ \boldsymbol{dx}$$

$$\text{Here } \frac{x^2 + x + 1}{(x + 1)^2 \ (x + 2)} = \frac{A}{(x + 1)} + \frac{B}{(x + 1)^2} + \frac{C}{x + 2}$$

$$\Rightarrow x^2 + x + 1 = A(x + 1)\ (x + 2) + B(x + 2) + C(x + 1)^2$$

On comparing, we get

A = – 2, B = 1 and C = 3.

$$\therefore I = \int \frac{-2 \ dx}{x+1} + \int \frac{dx}{(x + 1)^2} + 3 \int \frac{dx}{x + 2}$$

$$= -2 \log |x + 1| - \frac{1}{x + 1} + 3 \log |x + 2| + C$$

prepared by : BALAJI KANCHI

c. Linear and quadratic factors :

1.

$$\int \frac{1}{x\,(x^2+4)}\,dx$$

Sol.

$$\text{Let } I=\int\frac{1}{x(x^2+4)}dx=\int\frac{1}{x^3\left(1+\frac{4}{x^2}\right)}dx$$

$$\text{Let } \quad 1+\frac{4}{x^2}=t\Rightarrow\frac{-8}{x^3}dx=dt$$

$$\therefore\ I=\frac{-1}{8}\int\frac{dt}{t}$$

$$=\frac{-1}{8}\log|t|+C$$

$$=\frac{-1}{8}log\left[1+\frac{4}{x^2}\right]+C$$

2.

Find : $\int\frac{1}{x(x^2-1)}\,dx.$

Sol.

$$I=\int\frac{dx}{x(x^2-1)}=\int\frac{dx}{x^3(1-\frac{1}{x^2})}=\frac{1}{2}\int\frac{\left(\frac{2}{x^3}\right)dx}{(1-\frac{1}{x^2})}$$

$$\text{Put } (1-\frac{1}{x^2})=t\Rightarrow\left(\frac{2}{x^3}\right)dx=dt$$

$$I=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\log|(1-\frac{1}{x^2})|+c \quad \text{OR } \frac{1}{2}\log|(\frac{x^2-1}{x^2})|+c$$

3.

Find :

$$\int \frac{2}{(1-x)(1+x^2)}\,dx$$

Sol.

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$\Rightarrow A = 1, B = 1, C = 1$

Hence, $I = \int \frac{2}{(1-x)(1-x^2)}\,dx = \int \left[\frac{1}{1-x} + \frac{x+1}{1+x^2}\right] dx$

$= \int \frac{1}{1-x}\,dx + \int \frac{x+1}{x^2+1}\,dx$

$= \int \frac{1}{1-x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$= -\log|1-x| + \frac{1}{2}\log(x^2+1) + \tan^{-1}(x) + C$

prepared by : **BALAJI KANCHI**

4.

Find :

$$\int \frac{x}{(x^2+1)(x-1)}\,dx$$

Sol.

$$I=\int \frac{x}{\left(x^2+1\right)\left(x-1\right)}dx$$

$$\text{Let }\frac{x}{\left(x^2+1\right)\left(x-1\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$\Rightarrow x=A\left(x^2+1\right)+\left(Bx+C\right)\left(x-1\right)$$

$$\Rightarrow A=\frac{1}{2},B=-\frac{1}{2},C=\frac{1}{2}$$

$$I=\frac{1}{2}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{dx}{x^2+1}$$

$$=\frac{1}{2}\log|x-1|-\frac{1}{4}\log\left|x^2+1\right|+\frac{1}{2}\tan^{-1}x+C$$

5.

Find :

$$\int \frac{x}{(x-1)(x^2+4)}\,dx$$

Sol.

$$\int\frac{x}{\left(x-1\right)\left(x^2+4\right)}dx=\frac{1}{5}\int\frac{1}{x-1}dx-\frac{1}{10}\int\frac{2x}{x^2+4}dx+\frac{4}{5}\int\frac{1}{x^2+4}dx,\quad\text{(By Partial Fractions)}$$

$$=\frac{1}{5}\log|x-1|-\frac{1}{10}\log\left(x^2+4\right)+\frac{2}{5}\tan^{-1}\frac{x}{2}+C$$

6.

Find : $\int \frac{5x}{(x+1)(x^2+9)}\, dx.$

Sol.

$$\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow A = -\frac{1}{2}, B = \frac{1}{2}, C = \frac{9}{2}$$

Given integral

$$= -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{x+9}{x^2+9}dx$$

$$= -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{1}{4}\int \frac{18}{x^2+9}dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

7.

Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}\, dx$

Sol.

$$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+c}{x^2+1}$$

Getting $A = \frac{3}{5}, B = \frac{2}{5}, C = \frac{1}{5}$

Given integral $= \frac{3}{5}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log(x^2+1) + \frac{1}{5}tan^{-1}x + C$$

prepared by : BALAJI KANCHI

d. Two Quadratic factors :

1.

Evaluate :

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

Sol.

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)} = -\frac{4}{5}\frac{1}{t+4} + \frac{9}{5}\frac{1}{t+9}$$

$$= -\frac{4}{5}\frac{1}{(x^2+4)} + \frac{9}{5}\frac{1}{(x^2+9)}$$

$$\therefore \int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{4}{5}\int \frac{dx}{x^2+4} + \frac{9}{5}\int \frac{dx}{x^2+9}$$

$$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$$

2.

Find : $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$

Sol. $\int \frac{x^2 dx}{(x^2+1)(3x^2+4)}$

Let $x^2 = t$

$$\frac{t}{(t+1)(3t+4)} = \frac{-1}{t+1} + \frac{4}{3t+4} \quad \text{(by Partial fraction)}$$

$$\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx = \int -\frac{1}{x^2+1}\,dx + \int \frac{4}{3x^2+4}\,dx$$

$$= -\tan^{-1}x + \frac{4}{3}\times\frac{\sqrt{3}}{2}\tan^{-1}\frac{\sqrt{3}x}{2} + C$$

$$= -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}x}{2} + C$$

3.

$$\int \frac{2x}{(x^2+1)(x^2+2)}dx$$

Sol.

$$\int \frac{2x}{(x^2+1)(x^2+2)}dx$$

Let $x^2 = t$, $2x\,dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+2)}dx = \int \frac{dt}{(t+1)(t+2)}$$

Getting $\dfrac{1}{(t+1)(t+2)} = \dfrac{1}{t+1} + \dfrac{-1}{t+2}$ (by Partial Fraction)

$$\int \frac{dt}{(t+1)(t+2)} = \int \frac{dt}{t+1} - \int \frac{dt}{t+2}$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left(x^2+1\right) - \log\left(x^2+2\right) + C \text{ or } \log\left(\frac{x^2+1}{x^2+2}\right) + C$$

4.

Find : $\displaystyle\int \frac{2x}{(x^2+1)(x^2-4)}\,dx.$

Sol.

$$I = \int \frac{2x}{(x^2+1)(x^2-4)}dx$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$I = \int \frac{1}{(t+1)(t-4)}dt = \frac{1}{5}\int \frac{dt}{t-4} - \frac{1}{5}\int \frac{dt}{t+1}$$

$$I = \frac{1}{5}\log|x^2-4| - \frac{1}{5}\log|x^2+1| + c \text{ or } \frac{1}{5}\log\left|\frac{x^2-4}{x^2+1}\right| + c$$

5.

Find :

$$\int \frac{2x}{(x^2+3)(x^2-5)}\,dx$$

Sol.

$$\text{Let } I = \int \frac{2x}{(x^2+3)(x^2-5)}\,dx$$

$$\text{Put } x^2 = t \Rightarrow 2x.\,dx = dt$$

$$\Rightarrow I = \int \frac{dt}{(t+3)(t-5)}$$

$$= \int \left(-\frac{1}{8\,(t+3)} + \frac{1}{8(t-5)}\right) dt$$

$$= \frac{1}{8}[\log|t-5| - \log|t+3|] + c$$

$$= \frac{1}{8}\log\left|\frac{x^2-5}{x^2+3}\right| + c$$

6.

Find :

$$\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$$

Sol.

$$I = \int \frac{2x}{(x^2+1)(x^2+3)}\,dx, \qquad (\text{Put } x^2 = t, 2xdx = dt)$$

$$= \int \left(\frac{\frac{1}{2}}{t+1} - \frac{\frac{1}{2}}{t+3}\right) dt$$

$$= \frac{1}{2}[\log(t+1) - \log(t+3)] + c$$

$$= \frac{1}{2}[\log(x^2+1) - \log(x^2+3)] + c \quad \text{or} \quad \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c$$

7.

Find :

$$\int \frac{x^2+1}{(x^2+2)(x^2+4)}dx$$

Sol.

$$I=\int\frac{x^2+1}{\left(x^2+2\right)\left(x^2+4\right)}dx$$

Let $x^2=y$, then $\frac{x^2+1}{\left(x^2+2\right)\left(x^2+4\right)}=\frac{y+1}{(y+2)(y+4)}$

Let $\frac{y+1}{(y+2)(y+4)}=\frac{A}{y+2}+\frac{B}{y+4}$

this gives $A=-\frac{1}{2}, B=\frac{3}{2}$

$$\therefore I=-\frac{1}{2}\int\frac{1}{x^2+2}dx+\frac{3}{2}\int\frac{1}{x^2+4}dx$$

$$\Rightarrow I=-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{3}{4}\tan^{-1}\left(\frac{x}{2}\right)+c$$

prepared by : BALAJI KANCHI

8.

Find :

$$\int\frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$$

Sol.

$$\int\frac{x^2+1}{\left(x^2+2\right)\left(2x^2+1\right)}dx=\frac{1}{3}\int\frac{1}{x^2+2}dx+\frac{1}{3}\int\frac{1}{2x^2+1}dx \quad \textbf{(Using Partial Fractions)}$$

$$=\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}x\right)+C$$

$$\text{or }=\frac{1}{3\sqrt{2}}\left(\tan^{-1}\frac{x}{\sqrt{2}}+\tan^{-1}\sqrt{2}x\right)+C$$

9.

Find : $\int \frac{x^3+x}{x^4-9}\, dx.$

Sol. $I=\int \frac{x^3+x}{x^4-9}dx=\int \frac{x^3}{x^4-9}dx+\int \frac{x}{x^4-9}dx$

$=\frac{1}{4}\log|x^4-9|+\frac{1}{2}\int \frac{dt}{t^2-3^2}$, where $x^2=t$

$=\frac{1}{4}\log|x^4-9|+\frac{1}{2}\left[\frac{1}{2(3)}\log\left|\frac{t-3}{t+3}\right|\right]+C$

$=\frac{1}{4}\log|x^4-9|+\frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|+C$

e. Improper partial fraction :

1.

Find :

$$\int \frac{x^2}{x^2+6x+12}\, dx$$

Sol.

Let $I=\int \frac{x^2}{x^2+6x+12}\, dx$

$=\int \left[1-\frac{6x+12}{x^2+6x+12}\right] dx$

$\int \left[1-\frac{6(x+2)}{x^2+6x+12}\right] dx$

$x-6\int \frac{x+2}{x^2+6x+12}\, dx$ ______________ (1)

Let $I_1=\int \frac{x+2}{x^2+6x+12}\, dx$

$$x + 2 = A\frac{d}{dx}(x^2 + 6x + 12) + B$$

$$\Rightarrow x + 2 = A(2x + 6) + B$$

$$\Rightarrow A = \frac{1}{2} \text{ and } B = -1$$

$$\text{So, } I_1 = \frac{1}{2}\int \frac{2x + 6}{x^2 + 6x + 12}\,dx - \int \frac{1}{x^2 + 6x + 12}\,dx$$

$$= \frac{1}{2}\log|x^2 + 6x + 12| - \int \frac{1}{(x + 3)^2 + (\sqrt{3})^2}\,dx$$

$$= \frac{1}{2}\log|x^2 + 6x + 12| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x + 3}{\sqrt{3}}\right) + C$$

From (1), we have

$$I = x - 3\log|x^2 + 6x + 12| + 2\sqrt{3}\tan^{-1}\left(\frac{x + 3}{\sqrt{3}}\right) + C$$

prepared by : BALAJI KANCHI

2.

Find :

$$\int \frac{x^3 - 1}{x^3 - x}\,dx$$

Sol.

$$\int \frac{x^3 - 1}{x^3 - x}\,dx = \int\left(1 + \frac{1}{x} - \frac{1}{x+1}\right)dx$$
$$= x + \log|x| - \log|x + 1| + c$$

3.

Find : $\int \frac{x^4}{(x-1)(x^2+1)}\, dx$

Sol.

$$I = \int \frac{x^4}{(x-1)(x^2+1)} dx = \int \left[x + 1 + \frac{1}{(x-1)(x^2+1)} \right] dx$$

$$= \frac{x^2}{2} + x + \int \left[\frac{1}{2(x-1)} - \frac{1}{2} \frac{(x+1)}{(x^2+1)} \right] dx$$

(Using partial fractions)

$$= \frac{x^2}{2} + x + \frac{1}{2} \log |x-1| - \frac{1}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C$$

f. After substitution partial fraction :

Trigonometric substitution :

1.

Find :

$$\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx$$

Sol.

$\int \frac{(3\cos x-2)\sin x}{5- sin^2 x-4\cos x} dx$, Put cos x = t so that, −sin x dx = dt

$$= \int \frac{2-3t}{5-(1-t^2)-4t} dt$$

$$= \int \frac{2-3t}{(t-2)^2} dt$$

$$\int \frac{2-3t}{(t-2)^2} dt = -3 \int \frac{1}{t-2} dt - 4 \int \frac{1}{(t-2)^2} dt$$

$$= -3 \log|t-2| - 4 \left(\frac{-1}{t-2}\right) + C$$

$$= -3 \log|\cos x - 2| + \frac{4}{\cos x - 2} + C$$

2.

Find :

$$\int \frac{\cos x}{(4+\sin^2 x)(5-4\cos^2 x)}\, dx$$

Sol.

$$I=\int \frac{\cos x}{\left(4+\sin^2 x\right)\left(5-4\cos^2 x\right)}dx$$

$$=\int \frac{\cos x}{\left(4+\sin^2 x\right)\left(1+4\sin^2 x\right)}dx$$

$\sin x = t$ gives

$$I=\int \frac{dt}{\left(4+t^2\right)\left(1+4t^2\right)}$$

$$=-\frac{1}{15}\int \frac{dt}{4+t^2}+\frac{4}{15}\int \frac{dt}{1+4t^2} \qquad (\because \text{using Partial Fraction})$$

$$=-\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right)+\frac{2}{15}\tan^{-1}(2t)+C$$

$$=-\frac{1}{30}\tan^{-1}\left(\frac{\sin x}{2}\right)+\frac{2}{15}\tan^{-1}(2\sin x)+C$$

prepared by : BALAJI KANCHI

@kanchibalaji7
+91-8099454846
@ikbmaths7

Exponential substitution :

1.

$$\int \frac{e^{x}}{(e^{x}+1)(e^{x}+3)}dx$$

$$I = \int \frac{e^{x}}{(e^{x}+1)(e^{x}+3)}dx$$

Put $e^{x} = t \Rightarrow e^{x}\,dx = dt$

$$I = \int \frac{dt}{(t+1)(t+3)}$$

$$= \frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t+3}\right)dt$$

$$= \frac{1}{2}[\log|t+1|-\log|t+3|]+C$$

$$= \frac{1}{2}[\log|e^{x}+1|-\log|e^{x}+3|]+C$$

$$\text{or } \frac{1}{2}\log\left|\frac{e^{x}+1}{e^{x}+3}\right|+C$$

prepared by : BALAJI KANCHI

Algebraic substitution :

1.

Find :

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)}\,dx$$ **(put $x=t^2$ for linear factors)**

Sol.

$$\text{Let } I = \int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)(\sqrt{x}+2)}$$

$$\text{Let } \sqrt{x} = t,\ \frac{1}{2\sqrt{x}}\,dx = dt$$

$$\therefore I = 2\int \frac{dt}{(t+1)(t+2)}$$

$$= 2\int\left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$

$$= 2[\log|t+1| - \log|t+2|] + C$$

$$= 2[\log(\sqrt{x}+1) - \log(\sqrt{x}+2)] + C \text{ or } 2\log\left(\frac{\sqrt{x}+1}{\sqrt{x}+2}\right) + C$$

2.

Find :

$$\int \frac{\sqrt{x}}{(x+1)(x-1)}dx$$

Sol.

$$I=\int\frac{\sqrt{x}}{(x+1)(x-1)}dx$$

$$\sqrt{x}=t \text{ gives } I=\int\frac{2t^2}{(t^2+1)(t^2-1)}dt$$

$$\text{Let } \frac{2t^2}{(t^2+1)(t^2-1)}=\frac{2z}{(z+1)(z-1)}\text{ ; where } t^2=z$$

$$\text{we have } \frac{2z}{(z+1)(z-1)}=\frac{1}{z+1}+\frac{1}{z-1}$$

$$\Rightarrow I=\int\frac{1}{t^2+1}dt+\int\frac{1}{t^2-1}dt$$

$$=\tan^{-1}t+\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+c$$

$$=\tan^{-1}\sqrt{x}+\frac{1}{2}\log\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|+c$$

prepared by : BALAJI KANCHI

3.

Find :

$$\int \frac{2x^2+1}{x^2(x^2+4)}\,dx$$

Sol.

$$I=\int \frac{2x^2+1}{x^2\left(x^2+4\right)}dx$$

Let $\dfrac{2x^2+1}{x^2\left(x^2+4\right)}=\dfrac{2y+1}{y\left(y+4\right)}$, where $x^2=y$

Put $\dfrac{2y+1}{y\left(y+4\right)}=\dfrac{A}{y}+\dfrac{B}{y+4}$

$\Rightarrow 2y+1=A\left(y+4\right)+By$

$\Rightarrow A=\dfrac{1}{4}, B=\dfrac{7}{4}$

$\therefore \dfrac{2y+1}{y\left(y+4\right)}=\dfrac{1}{4y}+\dfrac{7}{4\left(y+4\right)}=\dfrac{1}{4x^2}+\dfrac{7}{4\left(x^2+4\right)}$

$\Rightarrow I=\dfrac{1}{4}\displaystyle\int \frac{1}{x^2}dx+\frac{7}{4}\int \frac{1}{x^2+4}dx$

$=-\dfrac{1}{4x}+\dfrac{7}{8}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

prepared by : BALAJI KANCHI

4.

Find : $\int \frac{2x+3}{x^2(x+3)}\,dx$

Sol.

$$\text{Find} : \int \frac{2x+3}{x^2(x+3)}\,dx$$

$$I = \int \frac{2x+3}{x^2(x+3)}dx = \int \frac{x+3}{x^2(x+3)}dx + \int \frac{x}{x^2(x+3)}dx$$

$$I = \int \frac{1}{x^2}dx + \frac{1}{3}\int \frac{x+3-x}{x(x+3)}dx = \int \frac{1}{x^2}dx + \frac{1}{3}\int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+3}dx$$

$$I = \frac{-1}{x} + \frac{1}{3}\log|x| - \frac{1}{3}\log|x+3| + c$$

prepared by : BALAJI KANCHI

V. $\int f(x) * g(x)\, dx$ model :

$$\int f * g\, dx = f \int g - \int \left(f' \times \int g \right) dx$$

Direct $\int f(x) * g(x)\, dx$ model :

1.

Find :

$$\int \sec^3 x\, dx$$

Sol.

$$I = \int \sec^3 \theta\, d\theta = \int \sec^2 \theta . \sec\theta\, d\theta$$

$$I = \sec\theta \int \sec^2 \theta\, d\theta - \int \left(\frac{d(\sec\theta)}{d\theta} \right) \left(\int \sec^2 \theta\, d\theta \right) d\theta$$

$$I = \sec\theta \tan\theta - \int \sec^3 \theta\, d\theta + \int \sec\theta\, d\theta$$

$$I = \frac{1}{2}\left(\sec\theta \tan\theta + \log|\sec\theta + \tan\theta| + c \right)$$

prepared by : BALAJI KANCHI

2.

$$\int e^x . \sin 2x\, dx$$

$$I = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin 2x}\, dx$$

$$I = \sin 2x\, e^x - \int \underset{\text{I}}{2\cos 2x}\, \underset{\text{II}}{e^x}\, dx$$

$$= e^x \sin 2x - 2[\cos 2x\, e^x - \int (-2\sin 2x) e^x dx]$$

$$I = e^x \sin 2x - 2\cos 2x\, e^x - 4I$$

$$5I = e^x \sin 2x - 2\cos 2x e^x$$

$$\therefore I = \frac{1}{5} e^x [(\sin 2x - 2\cos 2x)] + C$$

prepared by : BALAJI KANCHI

3.

$$\int \sin^{-1} x \, dx$$

Ans. $I = \int \sin^{-1} x \cdot 1 \, dx$

$$= \sin^{-1}x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x \, dx$$

$$= x \cdot \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx$$

$$= x \cdot \sin^{-1} x + \frac{1}{2} \cdot 2\sqrt{1-x^2} + C$$

or $x \sin^{-1}x + \sqrt{1-x^2} + C$

4.

Find :

$$\int x^2 \, . \, \sin^{-1}(x^{3/2})\, dx$$

Sol.

Let $x^{\frac{3}{2}} = t$

$\Rightarrow \frac{3}{2}x^{\frac{1}{2}}dx = dt$

The given integral becomes $\frac{2}{3}\int t\, sin^{-1}t dt$

$= \frac{2}{3}\left[sin^{-1}t \times \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \times \frac{t^2}{2} dt\right]$

$= \frac{1}{3}\left[sin^{-1}t \times t^2 + \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt\right]$

$= \frac{1}{3}\left[sin^{-1}t \times t^2 + \int \sqrt{1-t^2}\, dt - \int \frac{1}{\sqrt{1-t^2}} dt\right]$

$= \frac{1}{3}\left[t^2 sin^{-1}t + \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}sin^{-1}t - sin^{-1}t\right] + C$

$= \frac{1}{3}\left[t^2 sin^{-1}t + \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}sin^{-1}t\right] + C$

$= \frac{1}{3}\left[x^3 sin^{-1}\left(x^{\frac{3}{2}}\right) + \frac{x^{\frac{3}{2}}}{2}\sqrt{1-x^3} - \frac{1}{2}sin^{-1}\left(x^{\frac{3}{2}}\right)\right] + C$

5.

Find :

$$\int x^2 \log (x^2 + 1)\ dx$$

Sol.

$$\text{Let } I = \int x^2 \log (x^2 + 1) dx$$

$$= \log (x^2 + 1) . \frac{x^3}{3} - \int \frac{2x}{x^2 + 1} . \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \log (x^2 + 1) - \frac{2}{3} \int \frac{x^4}{x^2 + 1} dx$$

$$= \frac{x^3}{3} \log (x^2 + 1) - \frac{2}{3} \int \left(x^2 - 1 + \frac{1}{x^2 + 1}\right) dx$$

$$= \frac{x^3}{3} \log (x^2 + 1) - \frac{2}{3} \left[\frac{x^3}{3} - x + \tan^{-1} x\right] + C$$

prepared by : BALAJI KANCHI

6.

Find :

$$\int x^2 \log (x^2 - 1)\ dx$$

Sol.

$$\text{Given integral} = \log(x^2 - 1) \times \frac{x^3}{3} - \int \frac{2x}{x^2 - 1} \times \frac{x^3}{3} dx$$

$$= \log(x^2 - 1) \times \frac{x^3}{3} - \frac{2}{3} \int \frac{x^4 - 1 + 1}{x^2 - 1} dx$$

$$= \log(x^2 - 1) \times \frac{x^3}{3} - \frac{2}{3} \left[\int (x^2 + 1) dx + \int \frac{1}{x^2 - 1} dx\right]$$

$$= \log(x^2 - 1) \times \frac{x^3}{3} - \frac{2}{3} \left[\frac{x^3}{3} + x + \frac{1}{2} log \left|\frac{x - 1}{x + 1}\right|\right] + C$$

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7.

Find : $\int \frac{x+\sin x}{1+\cos x}\, dx$

Sol.

$$\int \frac{x+\sin x}{1+\cos x} dx$$

$$= \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} dx$$

$$= \int x\left(\frac{1}{2}\sec^2\frac{x}{2}\right) dx + \int \tan\frac{x}{2} dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2} dx + \int \tan\frac{x}{2} dx$$

$$= x\tan\frac{x}{2} + C$$

After substitution $\int f(x) * g(x)\, dx$ model :

1.

Find :

$$\int \cos x \,.\, \tan^{-1}(\sin x)\, dx$$

Ans. $I = \int \cos x \cdot \tan^{-1}(\sin x)\, dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore\ I = \int \tan^{-1} t \cdot 1\, dt$$

$$= \tan^{-1} t \cdot t - \frac{1}{2}\int \frac{2t}{1+t^2}\, dt$$

$$= t \cdot \tan^{-1} t - \frac{1}{2}\log|1+t^2| + C$$

$$= \sin x \cdot \tan^{-1}(\sin x) - \frac{1}{2}\log|1+\sin^2 x| + C$$

2.

Find : $\int 2x^3 e^{x^2}\, dx.$

Sol.

$I = \int 2x^3 e^{x^2} dx = \int 2x x^2 e^{x^2} dx$

Put $x^2 = t \Rightarrow 2x dx = dt$

$$I = \int t e^t dt$$

$$= te^t - \int e^t dt$$

$$= te^t - e^t + C$$

$$= e^{x^2}(x^2 - 1) + C$$

VI. $\int e^x(f + f')\,dx = e^x f(x) +$ model :

b. Trigometric functions :

1.

Find $\int e^{\cot^{-1}x}\left(\frac{1-x+x^2}{1+x^2}\right)dx.$

Sol.

Put $\cot^{-1}x = t \therefore x = \cot t$ and $\frac{1}{1+x^2}dx = -\,dt$

$$\therefore \int e^{\cot^{-1}x}\left(\frac{1-x+x^2}{1+x^2}\right)dx = -\int e^t\left(1-\cot t+\cot^2 t\right)dt = \int e^t\left(\cot t - \operatorname{cosec}^2 t\right)dt$$

$$= e^t \cot t + c$$

$$= xe^{\cot^{-1}x} + c$$

2.

Find $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Sol.

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx$$

$$= -e^x\cot\frac{x}{2} + c$$

3.

Find :

$$\int \frac{2+\sin 2x}{1+\cos 2x} e^x \, dx$$

Sol.

$$I=\int \frac{2+\sin 2x}{1+\cos 2x} e^x \, dx$$

$$=\int \frac{2+2\sin x \cos x}{2\cos^2 x} e^x \, dx$$

$$=\int \left(\sec^2 x+\tan x\right) e^x dx$$

$$=e^x . \tan x + c$$

prepared by : BALAJI KANCHI

c. Algebraic function:

1.

Find :

$$\int \frac{x-5}{(x-3)^3} e^x \, dx$$

Sol.

$$\int e^x \frac{x-5}{(x-2)^4} dx = \int e^x \left[\frac{(x-2)-3}{(x-2)^4}\right] dx$$

$$= \int e^x \left[\frac{1}{(x-2)^3} - \frac{3}{(x-2)^4}\right] dx$$

$$= \frac{e^x}{(x-2)^3} + C$$

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2.

Find : $\int e^x \left[\frac{1}{(1+x^2)^{\frac{3}{2}}} + \frac{x}{\sqrt{1+x^2}}\right] dx$

Sol.

$$I = \int e^x \left(\frac{x}{\sqrt{1+x^2}} + \frac{1}{(1+x^2)^{\frac{3}{2}}}\right)dx$$

$$\text{Let } f(x) = \frac{x}{\sqrt{1+x^2}}, f'(x) = \frac{\sqrt{1+x^2} - x\frac{x}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

On applying $\int e^x (f(x) + f'(x))dx = e^x f(x) + c,$

$$I = e^x \frac{x}{\sqrt{1+x^2}} + c$$


@kanchibalaji7
+91-8099454846
@ikbmaths7

d. Logarithm substitution
(logx=t then $\int e^x(f + f')\,dx$ form) :

1.a

Find : $\int e^{x^2}(x^5 + 2x^3)\,dx$

Sol.

Let $I = \int e^{x^2}(x^5 + 2x^3)\,dx$

Put $x^2 = t$ so that $2x\,dx = dt$

$\therefore I = \frac{1}{2}\int e^t(t^2 + 2t)\,dt$

$= \frac{1}{2}e^t t^2 + C$

$= \frac{1}{2}e^{x^2}(x^4) + C$

2.

Find : $\int \frac{\log x - 3}{(\log x)^4}\,dx.$

Sol. Put $x = e^t$ or $\log x = t$ $\quad dx = e^t dt$

$$\therefore \int \frac{\log x - 3}{(\log x)^4}dx = \int \frac{t-3}{t^4}e^t\,dt$$

$$= \int\left(\frac{1}{t^3} - \frac{3}{t^4}\right)e^t\,dt$$

$$= \frac{e^t}{t^3} + C$$

$$= \frac{x}{(\log x)^3} + C$$

Misc problems:

1.

Find :

$$\int \frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^2}\,dx$$

Due to printing error, the given function is not integrable.

So full marks may be given for every attempt.