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7.2 Definite Integrals

(Class XII CBSE Board Exam Models from 2022-2025 with solutions)

I. Definite Integrals :

a. general integration :

1.

Find :

$$\int_{1}^{4} \frac{1}{\sqrt{2x+1}-\sqrt{2x-1}}\, dx$$

Sol.

$$\text{Let } I = \int_1^4 \frac{1}{\sqrt{2x+1}-\sqrt{2x-1}} dx$$

$$= \int_1^4 \frac{\sqrt{2x+1}+\sqrt{(2x-1)}}{2} dx$$

$$= \frac{(2x+1)^{3/2}}{3 \times 2} + \frac{(2x-1)^{3/2}}{3 \times 2}\Bigg|_1^4$$

$$= \left(\frac{27}{6} + \frac{7^{3/2}}{6}\right) - \left(\frac{3^{3/2}}{6} + \frac{1}{6}\right)$$

$$= \frac{26 + 7^{3/2} - 3^{3/2}}{6} \text{ or } \frac{26 + 7\sqrt{7} - 3\sqrt{3}}{6}$$

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2.

Evaluate :

$$\int_0^{\pi/4} \frac{dx}{1+\tan x}$$

$$I = \int_0^{\pi/4} \frac{dx}{1+\tan x}$$

$$I = \int_0^{\pi/4} \frac{dx}{1+\tan\left(\frac{\pi}{4}-x\right)} \qquad \text{(using property)}$$

$$I = \int_0^{\pi/4} \frac{dx}{1+\frac{1-\tan x}{1+\tan x}}$$

$$I = \frac{1}{2}\int_0^{\pi/4} (1+\tan x)\,dx$$

$$I = \frac{1}{2}\left[x + \log \sec x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \log\sqrt{2}\right] = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}\log 2$$

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3.

If $\int_{a}^{b} x^3 \, dx = 0$ and $\int_{a}^{b} x^2 \, dx = \frac{2}{3}$, then find the values of a and b.

Sol.

$\int_{a}^{b} x^3 dx = 0 \Rightarrow \frac{b^4 - a^4}{4} = 0$

$\Rightarrow b^4 - a^4 = 0$

$\Rightarrow a = -b \quad (a \neq b)$

$\int_{a}^{b} x^2 dx = \frac{2}{3} \Rightarrow \frac{b^3 - a^3}{3} = \frac{2}{3}$

$\Rightarrow b^3 - a^3 = 2$

$\Rightarrow b^3 = 1$

$\Rightarrow b = 1$

$\Rightarrow b = 1, a = -1$

b.Based on trigonometric formulas :

1.

$$\text{Evaluate : } \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2x}{1+\cos 2x}\, dx$$

Sol.

$$I = \int_{-\pi/4}^{\pi/4} \frac{\cos 2x}{1+\cos 2x}\, dx = 2\int_{0}^{\pi/4} \frac{\cos 2x}{1+\cos 2x}\, dx$$

$$= 2\int_0^{\pi/4}\left(1 - \frac{1}{1+\cos 2x}\right)\, dx$$

$$= 2\int_0^{\pi/4}\left(1 - \frac{1}{2\cos^2 x}\right)\, dx$$

$$= 2\int_0^{\pi/4}\left(1 - \frac{1}{2}\sec^2 x\right)\, dx$$

$$= (2x - \tan x)\Big|_0^{\pi/4}$$

$$= \left(\frac{\pi}{2} - 1\right)$$

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2.

Find :

$$\int_{0}^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}}$$

Sol.

$$\int_{0}^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$$

$$=\frac{1}{2}\int_{0}^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}}$$

$$=\frac{1}{2}\int_{0}^{\pi/4} \frac{\left(1+\tan^2 x\right)\sec^2 x}{\sqrt{\tan x}}dx$$

$$\text{Put} \tan x = t \Rightarrow \sec^2 x\,dx = dt$$

$$\therefore I=\frac{1}{2}\int_{0}^{1} \frac{1+t^2}{\sqrt{t}}dt$$

$$=\frac{1}{2}\int_{0}^{1}\left(\frac{1}{\sqrt{t}}+t^{3/2}\right)dt$$

$$=\frac{1}{2}\left[2\sqrt{t}+\frac{2}{5}t^{5/2}\right]_0^1$$

$$=\frac{6}{5}$$

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3.

Evaluate :

$$\int_0^{\pi/2} \sin 2x \cos 3x \, dx$$

Sol.

$$I = \int_0^{\frac{\pi}{2}} \sin 2x \cos 3x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 5x - \sin x)\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{5}$$

4. 2025

Evaluate : $\displaystyle\int_0^{\pi} \frac{\sin 2px}{\sin x}\, dx,\ p \in N.$

Sol.

$$I = \int_0^{\pi} \frac{\sin 2px}{\sin x} dx$$

$$= \int_0^{\pi} \frac{\sin 2p(\pi - x)}{\sin (\pi - x)} dx$$

$$I = \int_0^{\pi} \frac{-\sin 2px}{\sin x} dx$$

Adding, we get

$2I = 0$

$\therefore I = 0$

c. $\int e^{x}(f + f')\,dx$ model :

1.

Evaluate :

$$\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$$

Sol.

Put 2x = t so that 2 dx = dt

When $x = \frac{\pi}{2}$, $t = \pi$, $x = \frac{\pi}{4}$ $t = \frac{\pi}{2}$

Thus

$$I = \int_{\pi/2}^{\pi} e^{t}\left(\frac{1-\sin t}{1-\cos t}\right)\frac{dt}{2}$$

$$= \int_{\pi/2}^{\pi} e^{t}\left(\frac{1-2\sin t/2\cos t/2}{2\sin^{2} t/2}\right)\frac{dt}{2}$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^{t}\left(\frac{1}{2}cosec^{2}\frac{t}{2} - \cot\frac{t}{2}\right)dt$$

$$= -\frac{1}{2}\left| e^{t}\cot\frac{t}{2}\right|_{\pi/2}^{\pi}$$

$$= -\frac{1}{2}\left| e^{\pi}\cot\frac{\pi}{2} - e^{\pi/2}\cot\frac{\pi}{4}\right|$$

$$= \frac{1}{2}e^{\pi/2}$$

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2.

$$\int_{0}^{\pi/2} e^{x}\left(\frac{1+\sin x}{1+\cos x}\right)dx$$

Sol.

$$\int_{0}^{\frac{\pi}{2}} e^{x}\left(\frac{1+\sin x}{1+\cos x}\right)\, dx$$

$$= \int_{0}^{\frac{\pi}{2}} e^{x}\left(\frac{1+2\sin\frac{x}{2}cos\frac{x}{2}}{2cos^{2}\frac{x}{2}}\right)\, dx$$

$$= \int_{0}^{\frac{\pi}{2}} e^{x}\left(tan\frac{x}{2}+\ \frac{1}{2}\ sec^{2}\frac{x}{2}\right)dx$$

On applying $\int e^{x}\,[\,f(x)+f'(x)]dx =\ e^{x}\,f(x)+C$

$$= \left[e^{x}\tan\frac{x}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}}$$

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4.

Evaluate :

$$\int_{\pi/2}^{\pi} e^{x}\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Sol.

$$I=\int_{\frac{\pi}{2}}^{\pi} e^{x}\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^{x}\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^{2}\frac{x}{2}}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^{x}\left(\frac{1}{2}\cos ec^{2}\frac{x}{2}-\cot\frac{x}{2}\right)dx$$

$$=-\left[e^{x}\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=e^{\frac{\pi}{2}}$$

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d. $\int f(x) * g(x)\,dx$ model :

1.

Evaluate :

$$\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

Ans Let $I = \int e^x \sin x \, dx$

$$= e^x \sin x - \int \cos x \, e^x \, dx$$

$$= e^x \sin x - \cos x \, e^x - I$$

$$\therefore I = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\therefore \int_0^{\pi/2} e^x \sin x \, dx = \frac{1}{2} e^{\pi/2} + \frac{1}{2} \text{ or } \frac{1}{2}(e^{\pi/2} + 1)$$

2.

Evaluate :

$$\int_0^1 x^2 e^x \, dx$$

Sol.

$$\int_0^1 x^2 e^x dx = \left[x^2 e^x\right]_0^1 - \int_0^1 2x e^x dx$$

$$= \left[x^2 e^x - 2x e^x + 2e^x\right]_0^1$$

$$= e - 2$$

3.

Evaluate: $\int_{0}^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)dx$

Sol.

Let $I = \int_{0}^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx.$

$= \int_{0}^{\pi/2} 2 \sin x \cos x \tan^{-1}(\sin x)\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Thus, $I = 2\int_{0}^{1} t \tan^{-1} t\, dt$

$= 2\left[\left|\frac{t^2}{2} \tan^{-1} t\right|_0^1 - \int_0^1 \frac{1}{1+t^2} \cdot \frac{t^2}{2}\, dt\right]$

$= 2 \cdot \frac{1}{2} \cdot \frac{\pi}{4} - \int_{0}^{1} \frac{t^2}{1+t^2}\, dt$

$= \frac{\pi}{4} - \int_0^1 \left[1 - \frac{1}{1+t^2}\right] dt$

$= \frac{\pi}{4} - |\, t\, |_0^1 + \left|\tan^{-1} t\right|_0^1$

$= \frac{\pi}{4} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{2} - 1$

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4.

Evaluate :

$$\int_0^1 \tan^{-1} x \, dx$$

Sol. Consider $\int (\tan^{-1} x)\, dx = \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$

$$= x \tan^{-1} x - \frac{1}{2}\log(1+x^2)$$

$$\int_0^1 (\tan^{-1} x) dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

5.

Evaluate : $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1} (\sin x) \, dx$

Sol.

$$\textbf{Let } I = \int_0^{\frac{\pi}{2}} \sin 2x \ \tan^{-1}(\sin x) \, dx$$

Put sin x = t so that cos x dx = dt

$$I = 2 \int_0^1 t \ \tan^{-1} t \ dt$$

$$= 2 \left[\tan^{-1} t \left(\frac{t^2}{2}\right) - \frac{1}{2} \int \frac{t^2}{1+t^2} dt \right]_0^1$$

$$= 2 \left[\left(\frac{t^2}{2}\right) \tan^{-1} t - \frac{1}{2} t + \frac{1}{2} \tan^{-1} t \right]_0^1$$

$$= 2 \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi}{2} - 1$$

e. Substitution method:

1.

Evaluate :

$$\int_0^{\pi/2} \sqrt{\sin x}\, \cos^5 x\, dx$$

Sol.

$$I = \int_0^{\pi/2} \sqrt{sin\ x}\ .\ (1 - sin^2\ x)^2 \cos x\ dx$$

Put $\sin x = t^2$

$$= \int_0^1 \sqrt{t}\ (1 - t^2)^2\ dt = \int_0^1 (\sqrt{t} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}})\ dt$$

$$= \left.\frac{2t^{3/2}}{3} + \frac{2t^{11/2}}{11} - \frac{4t^{7/2}}{7}\right]_0^1 = \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{64}{231}$$

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2.

Evaluate :

$$\int_0^{\frac{1}{2}\log 3} \frac{e^x}{e^{2x}+1}\, dx$$

Sol.

Let $e^x = t$, $e^x dx = dt$

$$\therefore \int_0^{\frac{1}{2}\log 3} \frac{e^x}{e^{2x}+1} dx = \int_1^{\sqrt{3}} \frac{dt}{t^2+1}$$

$$= \tan^{-1} t \Big|_1^{\sqrt{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

3.

Evaluate :

$$\int_1^e \frac{1}{\sqrt{4x^2-(x\log x)^2}}dx$$

Sol.

Let $I = \int_1^e \frac{1}{\sqrt{4x^2-(x\log x)^2}}dx$

$= \int_1^e \frac{1}{x\sqrt{4-(\log x)^2}}dx \qquad [\text{ Let } \log x = t \Rightarrow \frac{1}{x}dx = dt]$

$= \int_0^1 \frac{dt}{\sqrt{4-t^2}}$

$= \sin^{-1}\frac{t}{2}\Big|_0^1 = \frac{\pi}{6}$

4.

Evaluate :

$$\int_0^{a^3} \frac{x^2}{x^6+a^6}\, dx$$

Sol.

Put $x^3 = t \Rightarrow x^2 dx = \frac{dt}{3}$

Given integral $= \frac{1}{3}\int_0^{a^9} \frac{dt}{t^2+(a^3)^2}$

$= \frac{1}{3a^3}\tan^{-1}\frac{t}{a^3}\Big|_0^{a^9}$

$= \frac{1}{3a^3}\tan^{-1}a^6$

5.

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$$

Sol.

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx \qquad \text{Put } \sqrt{x} = t \ \Rightarrow\ dx = 2t\, dt$$

$$2\int_0^{\frac{\pi}{2}} \sin t\, dt = 2\ [-\cos t]_0^{\frac{\pi}{2}}$$

$$= 2$$

7.

Evaluate : $\displaystyle\int_{-2}^{1} \sqrt{5-4x-x^2}\, dx$

Sol.

$$\int_{-2}^{1} \sqrt{5-4x-x^2}\, dx$$

$$= \int_{-2}^{1} \sqrt{9-(x+2)^2}\, dx$$

$$= \frac{x+2}{2}\sqrt{9-(x+2)^2} + \frac{9}{2}\sin^{-1}\frac{x+2}{3}\Big]_{-2}^{1}$$

$$= 0 + \frac{9}{2}.\frac{\pi}{2} - 0$$

$$= \frac{9\pi}{4}$$

f. Partial fraction after Substitution :

1.

Evaluate : $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(4+\sin x)}\,dx.$

Let $\sin x = t$, then $\cos x dx = dt$

$$\therefore \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(4+\sin x)} dx = \int_0^1 \frac{dt}{(1+t)(4+t)}$$

$$= \frac{1}{3}\left[\int_0^1 \frac{1}{1+t}dt - \int_0^1 \frac{1}{4+t}dt\right]$$

$$= \frac{1}{3}\left[\log(1+t)\Big|_0^1 - \log(4+t)\Big|_0^1\right]$$

$$= \frac{1}{3}[\log 2 - \log 5 + \log 4]$$

or

$$= \frac{1}{3}\log\frac{8}{5}$$

2.

Evaluate $\displaystyle\int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x+e^{-x})(e^x-e^{-x})}\,dx$

$$\text{Let } I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x+e^{-x})(e^x-e^{-x})}dx = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{e^{2x}}{(e^{2x})^2-1}dx$$

Put $e^{2x} = t \Rightarrow e^{2x}dx = \frac{1}{2}dt$, Upper limit = 3, Lower Limit = 2

$$\therefore I = \frac{1}{2}\int_2^3 \frac{1}{t^2-1}dt = \frac{1}{4}\log\left|\frac{t-1}{t+1}\right|\Bigg]_2^3$$

$$\Rightarrow I = \frac{1}{4}\left[\log\frac{2}{4} - \log\frac{1}{3}\right] = \frac{1}{4}\log\frac{3}{2}$$

II. Based on properties

a. Apply Property : x→ $a+b-x$,
then add the integrals: common denominator ,
function will become one/denominator eliminates :

1.

Evaluate: $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$

Sol.

Let $I=\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$

$\Rightarrow I=\int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}\,dx=\int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx$

Adding (i) and (ii), we get

$2I=\int_0^{\pi} dx = x]_0^{\pi}=\pi, \quad \therefore I=\frac{\pi}{2}$

prepared by : BALAJI KANCHI

2

Evaluate :

$$\int_{-3}^{3} \frac{x^4}{1+e^x}\,dx$$

Sol.

Let, $I = \int_{-3}^{3} \frac{x^4}{1+e^x}\,dx \qquad \ldots(1)$

$I = \int_{-3}^{3} \frac{x^4}{1+e^{-x}}\,dx \qquad$ (using property)

$= \int_{-3}^{3} \frac{e^x x^4}{1+e^x}\,dx \qquad \ldots(2)$

Adding equations (1) and (2)

$$2I = \int_{-3}^{3} \frac{x^4(1+e^x)}{1+e^x}\,dx$$

$$\Rightarrow 2I = \left.\frac{x^5}{5}\right|_{-3}^{3}$$

$$\Rightarrow I = \frac{243}{5}$$

3.

Evaluate : $\displaystyle\int_0^{2\pi} \frac{dx}{1+e^{\sin x}}$

Sol.

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}} \qquad (1)$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin(2\pi - x)}}$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}}$$

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}}dx \qquad (2)$$

Adding (1) and (2)

$$2I = \int_0^{2\pi} 1\,dx \Rightarrow 2I = 2\pi$$

$$I = \pi$$

4.

Evaluate $\displaystyle\int_{-a}^{a} f(x)\,dx$, where $f(x) = \dfrac{9^x}{1+9^x}$.

Sol.

$$I = \int_{-a}^{a} \frac{9^x}{1+9^x}dx \text{ --------(i)}$$

Replacing 'x' by $-a+a-x$

$$I = \int_{-a}^{a} \frac{9^{a-a-x}}{1+9^{a-a-x}}dx = \int_{-a}^{a} \frac{9^{-x}}{1+9^{-x}}dx = \int_{-a}^{a} \frac{1}{1+9^{x}}dx \text{ --------(ii)}$$

Adding (i) & (ii) we get, $2I = \int_{-a}^{a} dx = x\Big]_{-a}^{a} = 2a \Rightarrow I = a$

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5.

Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x} dx$.

Sol.

$$I = \int_{-2}^{2} \frac{x^2}{1+5^x} \ dx \quad \text{-----------} \ (1)$$

Replacing x by -2+2-x

$$= \int_{-2}^{2} \frac{(-2+2-x)^2}{1+5^{(-2+2-x)}} \ dx$$

$$I = \int_{-2}^{2} \frac{x^2 5^x}{1+5^x} \ dx \ \text{------------}(2)$$

Adding (1) + (2)

$$2I = \int_{-2}^{2} \frac{x^2(1+5^x)}{1+5^x} dx = \int_{-2}^{2} x^2 \ dx = \left|\frac{x^3}{3}\right|_{-2}^{2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

$$\Rightarrow I = \frac{16}{6} \text{ or } \frac{8}{3}$$

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b. apply Property : x→ $a+b-x$, add the integrals then standard integration :

1.

Evaluate :

$$\int_{-2}^{2}\sqrt{\frac{2-x}{2+x}}\,dx$$

Sol.

$$\int_{-2}^{2}\sqrt{\frac{2-x}{2+x}}\,dx$$

$$=\int_{-2}^{2}\frac{2-x}{\sqrt{4-x^2}}\,dx$$

$$=\int_{-2}^{2}\frac{2}{\sqrt{4-x^2}}\,dx-\int_{-2}^{2}\frac{x}{\sqrt{4-x^2}}\,dx$$

$$=2\int_{0}^{2}\frac{2}{\sqrt{4-x^2}}\,dx-0\ \left[\frac{2}{\sqrt{4-x^2}}\text{is even, }\frac{x}{\sqrt{4-x^2}}\text{ is odd}\right]$$

$$=4\int_{0}^{2}\frac{1}{\sqrt{4-x^2}}\,dx$$

$$=4\sin^{-1}\frac{x}{2}\Big|_{0}^{2}$$

$$=2\pi$$

2.

Find the value of $\int_0^1 x(1-x)^n \, dx$.

Sol.

$$I = \int_0^1 x(1-x)^n \, dx$$

$$= \int_0^1 (1-x)[1-(1-x)]^n \, dx \quad \text{[using property]}$$

$$= \int_0^1 x^n (1-x) \, dx$$

$$= \int_0^1 x^n dx - \int_0^1 x^{n+1} dx$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} \quad \text{Or} \quad \frac{1}{(n+1)(n+2)}$$

3.

$$\int_0^5 x \cdot \sqrt{5-x} \, dx$$

Ans. $I = \int_0^5 x \sqrt{5-x} \, dx$

$$= \int_0^5 (5-x)\sqrt{x} \, dx$$

$$= \int_0^5 (5\sqrt{x} - x^{3/2}) \, dx$$

$$= 5 \times \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \Big]_0^5$$

$$= \frac{10}{3} \times 5\sqrt{5} - \frac{2}{5} \cdot 25\sqrt{5}$$

$$= \frac{20\sqrt{5}}{3}$$

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4.

Evaluate : $\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}\,dx$

Sol.

$$I=\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}\,dx$$

$$=\int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}}\,dx \qquad \text{(using property)}$$

$$\Rightarrow 2I=\int_1^3 1\,dx = x\big]_1^3 = 2$$

$$\Rightarrow I=1$$

5.

Evaluate :

$$\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\,dx$$

Sol.

$$\text{Let } I=\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\,dx$$

$$I=\int_1^3 \frac{\sqrt{4-(4-x)}}{\sqrt{4-x}+\sqrt{4-(4-x)}}\,dx$$

$$2I=\int_1^3 \frac{\sqrt{4-x}+\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}\,dx=\int_1^3 1\,dx$$

$$2I=x|_1^3=2$$

$$\Rightarrow I=1$$

6.

Evaluate, using properties :

$$\int_{-\pi}^{\pi} (3 \sin x - 2)^2 \, dx.$$

$$I = \int_{-\pi}^{\pi} (9\sin^2 x - 12\sin x + 4)\, dx$$

$$= \int_{-\pi}^{\pi} 9\sin^2 x\, dx - \int_{-\pi}^{\pi} 12\sin x\, dx + \int_{-\pi}^{\pi} 4\, dx$$

$$= 2\int_{0}^{\pi} 9\sin^2 x\, dx - 0 + 2\int_{0}^{\pi} 4\, dx$$

(As $9\sin^2 x$ and 4 are even and $12\sin x$ is odd)

$$= 9\int_{0}^{\pi} (1 - \cos 2x)\, dx + 8x \Big|_{0}^{\pi}$$

$$= 9\left(x - \frac{\sin 2x}{2}\right)\Bigg|_{0}^{\pi} + 8\pi$$

$$= 9\pi + 8\pi = 17\pi$$

c. apply Property : x→ $a+b-x$ for Logarthimic function :

1.

Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx$.

Sol.

$$\text{Let } I=\int_0^{\pi/4}\log(1+\tan x)\,dx \qquad \text{---------- (i)}$$

$$\Rightarrow I=\int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx \qquad \text{---------- (ii)}$$

$$\text{Add (i) and (ii), } 2I=\int_0^{\pi/4}\log 2\,dx=\log 2\cdot x\Big]_0^{\pi/4}=\frac{\pi}{4}\log 2$$

$$\Rightarrow I=\frac{\pi}{8}\log 2$$

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OR

$$I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx$$

$$=\int_0^{\frac{\pi}{4}}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx=\int_0^{\frac{\pi}{4}}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\log(2)\,dx-\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx$$

$$\Rightarrow I=\log 2\,[x]_0^{\pi/4}-I$$

$$\Rightarrow I=\frac{\pi}{8}\log 2$$

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2.

Evaluate :

$$\int_0^{\pi/2} (2 \log \cos x - \log \sin 2x)\, dx$$

Sol.

$$\int_0^{\frac{\pi}{2}} (2\log\cos x - \log\sin 2x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\cos^2 x}{2\sin x\cos x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot x}{2}\right)dx = I \text{ (say)} \qquad \ldots (1)$$

$$I = \int_0^{\frac{\pi}{2}} \log\frac{\tan x}{2}\,dx \qquad \text{(using property)} \qquad \ldots (2)$$

Adding (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\frac{\cot x}{2}\right)dx$$

$$2I = \log\left(\frac{1}{4}\right)[x]_0^{\pi/2}$$

$$I = \frac{\pi}{4}\log\frac{1}{4} \quad or \quad -\frac{\pi}{2}\log 2$$

prepared by : BALAJI KANCHI

3.

Evaluate :

$$\int_0^{\frac{\pi}{2}} [\log(\sin x) - \log(2\cos x)]\,dx.$$

Ans Let $I = \int_0^{\pi/2} [\log \sin x - \log(2\cos x)]dx = \int_0^{\pi/2} \log\left(\frac{\tan\ x}{2}\right)dx$

Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

We get, $I = \int_0^{\pi/2} \log\left(\frac{\cot\ x}{2}\right)dx$

$\therefore\ 2I = \int_0^{\pi/2} \log\left(\frac{\tan\ x}{2} \times \frac{\cot\ x}{2}\right)dx = \int_0^{\pi/2} \log\left(\frac{1}{4}\right)dx$

$2I = \log\left(\frac{1}{4}\right)x\Big|_0^{\pi/2} = \frac{\pi}{2}\log\frac{1}{4}$

$I = \frac{\pi}{4}\ log\ \frac{1}{4}$ OR $-\frac{\pi}{2}\ log\ \ 2$

4.

Evaluate :

$$\int_1^2 \log\left(\frac{3}{x}-1\right)dx$$

Ans. $I = \int_1^2 \log\left(\frac{3}{x}-1\right)dx$

$= \int_1^2 [\log(3-x) - \log x]\,dx$

$= \int_1^2 \log(3-x)\,dx - \int_1^2 \log x\,dx$

$= \int_1^2 \log(3-x)\,dx - \int_1^2 \log(3-x)\,dx$ $\quad (\therefore \text{ Using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$= 0$

prepared by : BALAJI KANCHI

d. Property : x→ $\frac{\pi}{2} - x$, if the function in terms of sinx , cosx. then add integrals , common denominator , function will become one /denominator eliminates :

1.

Evaluate :

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx$$

Sol.

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx$$

$$I = 2\int_{0}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx \quad \text{as } f(x) = \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} \text{ is even}$$

$$I = 2\int_{0}^{\pi/2} \frac{\cos^{100} x}{\cos^{100} x + \sin^{100} x} dx \quad \text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$2I = 2\int_{0}^{\pi/2} \frac{\sin^{100} x + \cos^{100} x}{\cos^{100} x + \sin^{100} x} dx = 2\int_0^{\pi/2} dx$$

$$I = x|_0^{\pi/2} \quad \Rightarrow I = \frac{\pi}{2}$$

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2.

Evaluate :

$$\int_0^{\pi/2} \frac{1}{1+\cot^{5/2}x}dx$$

Ans. $I = \int_0^{\pi/2} \frac{1}{1+\cot^{5/2}x}dx = \int_0^{\pi/2} \frac{\sin^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$ ...(i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$ ...(ii)

adding (i) and (ii)

$$2I = \int_0^{\pi/2} 1\cdot dx \Rightarrow 2I = x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

prepared by : BALAJI KANCHI

3.

Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$.

Sol.

Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Using property = $\int_a^b f(x)dx == \int_a^b f(a+b-x)dx$ we get

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

Adding we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_{\pi/6}^{\pi/3} 1dx$$

$$= x|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

4.

Evaluate :

$$\int_0^{\pi/2} \frac{1}{1+(\tan x)^{2/3}} dx$$

Sol.

$$I = \int_0^{\pi/2} \frac{dx}{1+(\tan x)^{2/3}}$$

$$= \int_0^{\pi/2} \frac{\cos^{2/3} x}{\cos^{2/3} x + \sin^{2/3} x} dx \quad \ldots \text{(I)}$$

$$= \int_0^{\pi/2} \frac{\cos^{2/3}\left(\frac{\pi}{2} - x\right)}{\cos^{2/3}\left(\frac{\pi}{2} - x\right) + \sin^{2/3}\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sin^{2/3} x}{\sin^{2/3} x + \cos^{2/3} x} dx \quad \ldots \text{(II)}$$

(I) + (II)

$$\Rightarrow 2I = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

5.

Evaluate : $\int_{0}^{\frac{\pi}{2}} \frac{5\sin x + 3\cos x}{\sin x + \cos x}\, dx$

Sol.

$$I = \int_{0}^{\pi/2} \frac{5\sin x + 3\cos x}{\sin x + \cos x} dx \qquad \text{-- (i)}$$

$$I = \int_{0}^{\pi/2} \frac{5\sin\left(\frac{\pi}{2} - x\right) + 3\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_{0}^{\pi/2} \frac{5\cos x + 3\sin x}{\cos x + \sin x} dx \qquad \text{-- (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\pi/2} 8dx \Rightarrow I = 4\,x\Big]_{0}^{\frac{\pi}{2}} = 2\pi$$

e. Apply the Property : x→ $a-x$ and add integrals, to eliminate variable x in the numerator :

(or) apply the property : $\int_a^b x\ f(x)\,dx = \frac{a+b}{2}\int_a^b f(x)\ dx$

1.

Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Sol.

$$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \qquad ...(i)$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \qquad ...(ii)$$

Adding (i) and (ii)

$$2I=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x=t \Rightarrow -\sin x\,dx=dt$

$$I=-\frac{\pi}{2}\int_1^{-1}\frac{dt}{1+t^2}$$

$$=\pi\left[\tan^{-1}t\right]_0^1=\frac{\pi^2}{4}$$

2.

Evaluate :

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

Sol.

$$\text{Let } I=\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \Rightarrow I=\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Substituting, $\cos x = t, \sin x dx = -dt$, we get,

$$I=\frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2}=-\frac{\pi}{2}\tan^{-1} t\Big]_1^{-1}$$

$$=-\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=\frac{\pi^2}{4}$$

3.

Evaluate :

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$$

Sol.

$$\text{Let } I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \text{-- (i)}$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad \text{-- (ii)}$$

Adding (i) & (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx \Rightarrow 2I = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$$

$$= \pi\int_0^{\pi} (\sec x \tan x - \sec^2 x + 1) dx$$

$$= \pi(\sec x - \tan x + x)\Big]_0^{\pi}$$

$$= \pi(-1 + \pi - 1) = \pi(\pi - 2)$$

$$\therefore I = \frac{\pi}{2}(\pi - 2) \text{ or } \pi\left(\frac{\pi}{2} - 1\right)$$

prepared by : BALAJI KANCHI

4.

Evaluate :

$$\int_0^{\pi} \frac{x}{9\sin^2 x + 16\cos^2 x}\,dx$$

Let, $I = \int_0^{\pi} \frac{x}{9\sin^2 x + 16\cos^2 x}\,dx$ ...(1)

$I = \int_0^{\pi} \frac{\pi - x}{9\sin^2 x + 16\cos^2 x}\,dx$ (using property) ...(2)

Adding equation (1) and (2)

$$2I = \int_0^{\pi} \frac{\pi}{9\sin^2 x + 16\cos^2 x}\,dx$$

$\Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \frac{1}{9\sin^2 x + 16\cos^2 x}\,dx$ (using property)

$$I = \pi\left[\int_0^{\pi/4} \frac{\sec^2 x}{9\tan^2 x + 16}\,dx + \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2 x}{9 + 16\cot^2 x}\,dx\right] = \pi\left[I_1 + I_2\right] \text{ (say)}$$

$= \pi\left[\int_0^1 \frac{dt}{9t^2 + 16} - \int_1^0 \frac{dz}{9 + 16z^2}\right]$ $(t = \tan x \text{ in } I_1,\ z = \cot x \text{ in } I_2)$

$$= \frac{\pi}{12}\left\{\left[\tan^{-1}\frac{3t}{4}\right]_0^1 - \left[\tan^{-1}\frac{4z}{3}\right]_1^0\right\}$$

$$= \frac{\pi}{12}\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{4}{3}\right)$$

or $\frac{\pi}{12} \times \frac{\pi}{2} = \frac{\pi^2}{24}$

5.

Evaluate: $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

$$I=\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx \quad ...(1)$$

using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$I=\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x+\sin^4 x}dx \quad ...(2)$$

adding (1) and (2)

$$2I=\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\pi\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\frac{\pi}{4}} \frac{\tan x\sec^2 x}{\left(\tan^2 x\right)^2+1}dx \quad (\because \text{dividing by} \cos^4 x)$$

Putting $\tan^2 x=t$ gives $I=\frac{\pi}{4}\int_0^1 \frac{1}{t^2+1}dt$

$$\Rightarrow I=\frac{\pi^2}{16}$$

prepared by : **BALAJI KANCHI**

6.

Evaluate :

$$\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$$

Sol.

$$I = \int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$$

dividing numerator and denominator by $\cos^4 x$,

$$I = \int_0^{\pi/4} \frac{\tan x \sec^2 x}{1 + \tan^4 x}\, dx$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$

when $x = 0, t = 0$; when $x = \frac{\pi}{4}, t = 1$

$$\Rightarrow I = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$= \frac{\pi}{8}$$

7.

Evaluate :

$$\int_{0}^{\pi/2} \frac{x}{\sin x + \cos x}\, dx$$

Sol.

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{x}{\text{sinx+cosx}}\, dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\text{cosx+sinx}}\, dx \qquad \text{using property}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{\text{sinx+cosx}}\, dx$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(\frac{\pi}{4} + x\right)}\, dx$$

$$2I = \frac{\pi}{2\sqrt{2}} \log \left|\text{cosec}\left(\frac{\pi}{4} + x\right) - \cot\left(\frac{\pi}{4} + x\right)\right|_0^{\frac{\pi}{2}}$$

$$I = \frac{\pi}{4\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

OR

Sol.

$$I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}}} dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$I = \frac{\pi}{4\sqrt{2}} log \left| sec\left(x - \frac{\pi}{4}\right) + tan\left(x - \frac{\pi}{2}\right) \right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4\sqrt{2}} \left[log(\sqrt{2} + 1) - log(\sqrt{2} - 1) \right]$$

$$\text{Or } I = \frac{\pi}{4\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

8.

Evaluate : $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

Sol. $I = \int_0^{\pi} \frac{x}{1+\sin x} dx = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x} dx$

$\therefore\ 2I = \pi \int_0^{\pi} \frac{dx}{1+\sin x}$

$\therefore\ I = \frac{\pi}{2} \int_0^{\pi} \frac{dx}{1+\sin x} = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos\left(\frac{\pi}{2} - x\right)} dx$

$= \frac{\pi}{2} \int_0^{\pi} \frac{1}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} dx$

$= \frac{\pi}{4} \left[-2\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]_0^{\pi}$

$= \frac{\pi}{4} [2 - (-2)] = \pi$

9.

Evaluate : $\int_{0}^{\frac{\pi}{4}} \frac{x\,dx}{1+\cos 2x+\sin 2x}$

Sol.

$I=\int_{0}^{\frac{\pi}{4}} \frac{x}{1+\cos 2x+\sin 2x}dx \quad ....(1)$

On applying $\int_{0}^{a} f(x)dx=\int_{0}^{a} f(a-x)dx,$

we get $I=\int_{0}^{\frac{\pi}{4}} \frac{\frac{\pi}{4}-x}{1+\cos 2x+\sin 2x}dx...(2)$

On adding Eq. (1) and (2), we get $2I=\frac{\pi}{4}\int_{0}^{\frac{\pi}{4}} \frac{1}{1+\cos 2x+\sin 2x}dx$

$I=\frac{\pi}{16}\int_{0}^{\frac{\pi}{4}} \frac{1}{\cos^2 x+\sin x\cos x}dx=\frac{\pi}{16}\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x dx}{1+\tan x}$

$I=\frac{\pi}{16}(\log|1+\tan x|)_0^{\frac{\pi}{4}}$

$I=\frac{\pi}{16}\log 2$

prepared by : BALAJI KANCHI

III. Using formulas

a. $1 + sin2x = (sinx + cosx)^2$,

$1 - sin2x = (sinx - cosx)^2$,

$sin2x = 1 - (sinx - cosx)^2$, then $sinx \pm cosx = t$

1.

$$\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Sol.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sinx+cosx}{\sqrt{sin2x}} dx$$

Put $sinx - cosx = t$ so that $(cosx + sinx)\, dx = dt$

$$= \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1- t^2}}$$

$$= [\sin^{-1} t]_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}}$$

$$= 2 \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

2.

Evaluate :

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx$$

Sol.

Put $\sin x - \cos x = t$, so that $(\cos x + \sin x)\, dx = dt$

$$\sin^2 x + \cos^2 x - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$$

$$I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16} \int_{-1}^{0} \frac{dt}{(\frac{5}{4})^2 - t^2}$$

$$= \frac{1}{40}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log 1 - \log\left(\frac{1}{9}\right)\right] = \frac{1}{40}\log 9 \ \ or\ \frac{1}{20}\log 3$$

prepared by : **BALAJI KANCHI**

b. $sinx + cosx = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ or $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$:

1.

Evaluate :

$$\int_{0}^{\pi/4} \frac{1}{\sin x + \cos x} dx$$

Sol.

$$I = \int_{0}^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{4}} \frac{1}{\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{4}} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)} dx = = \frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{4}} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \left[\log \left| \operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right) \right| \right]_{0}^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \log\left(\sqrt{2} + 1\right) \text{ or } -\frac{1}{\sqrt{2}} \log\left(\sqrt{2} - 1\right)$$

IV. If $\dfrac{1}{a+ b\cos^2 x}$, $\dfrac{1}{a\cos^2 x+b\sin^2 x}$, $\dfrac{1}{a+ b\sin^2 x}$ form

Then dividing every term by $\cos^2$x and converts denominator into $(a + b\,tan^2 x)$, numerator $sec^2 x$ term , put $tan\,x = t$ then quadratic polynomial form :

1.

Evaluate :

$$\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Sol.

$$I=\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$=2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$=2\int_0^{\pi/2}\frac{\sec^2 x}{a^2+b^2\tan^2 x}dx$$

$\tan x = t$ gives

$$I=2\int_0^{\infty}\frac{dt}{a^2+b^2t^2}$$

$$=\frac{2}{b^2}.\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\Bigg]_0^{\infty}$$

$$=\frac{\pi}{ab}$$

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VI.Modulus function Integration :

a. Trogonometric function :

1.

Evaluate :

$$\int_{0}^{3/2} |\, x\cos \pi x \,| \; dx$$

Sol.

$$I = \int_{0}^{3/2} |x\cos \pi x| dx$$

$$= \int_{0}^{1/2} x\cos \pi x \, dx - \int_{1/2}^{3/2} x\cos \pi x \, dx \quad \ldots(1)$$

$$\text{Consider} \int x\cos \pi x \, dx$$

$$= \frac{x\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx$$

$$= \frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \quad \ldots(2)$$

using (2) in (1),

$$\left. \frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_0^{1/2} - \left. \frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi} - \frac{1}{\pi^2} \right) - \left(-\frac{3}{2\pi} - \frac{1}{2\pi} \right)$$

$$= \frac{5}{2\pi} - \frac{1}{\pi^2}$$

2.

Evaluate :

$$\int_{0}^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$$

Sol.

$$\text{Let } I = \int_{0}^{\pi/2} |\sin x - \cos x|\, dx$$

$$= \int_{0}^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$$

$$= (\cos x + \sin x)\Big|_{0}^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) \quad - 1 + \sqrt{2}$$

$$= 2\sqrt{2} - 2$$

3.

Evaluate : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\sin |x| + \cos |x|)\, dx$

Sol. $I = \int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)dx$

$f(x) = \sin|x| + \cos|x|$

$f(x)$ is an even function

$$I = 2\int_{0}^{\pi/2} (\sin|x| + \cos|x|)dx$$

$$= 2\int_{0}^{\pi/2} (\sin x + \cos x)dx$$

$$= 2[[-\cos x]_{0}^{\pi/2} + [\sin x]_{0}^{\pi/2}]$$

$$= 2[1+1]$$

$$= 4$$

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b.Polynomial/Algebraic function :

1.

Evaluate $\int_{-1}^{1} |x^4 - x|\, dx.$

Sol.

$$I = \int_{-1}^{1} |x^4 - x| dx = \int_{-1}^{0} (x^4 - x) dx - \int_{0}^{1} (x^4 - x) dx$$

$$= \left(\frac{x^5}{5} - \frac{x^2}{3}\right)\Bigg]_{-1}^{0} - \left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Bigg]_{0}^{1}$$

$$= \frac{7}{10} + \frac{3}{10} = I$$

2.

Evaluate : $\int_{-1}^{2} |\ x^3 - x\ |\ dx$

Sol. Here, $x^3 - x \geq 0$ on $[-1, 0]$, $x^3 - x \leq 0$ on $[0, 1]$ and $x^3 - x \geq 0$ on $(1, 2)$

So, $I = \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} (x - x^3) dx + \int_{1}^{2} (x^3 - x) dx$

$$= \left.\frac{x^4}{4} - \frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2} - \frac{x^4}{4}\right|_{0}^{1} + \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{1}^{2}$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{9}{4}$$

$$= \frac{11}{4}$$

3.

Evaluate :

$$\int_{-4}^{0} |x+2| \, dx$$

Sol.

$$\int_{-4}^{0} |x+2| dx = \int_{-4}^{-2} -(x+2) dx + \int_{-2}^{0} (x+2) dx$$

$$= -\frac{(x+2)^2}{2}\Bigg]_{-4}^{-2} + \frac{(x+2)^2}{2}\Bigg]_{-2}^{0} = 2+2=4$$

4.

Evaluate :

$$\int_{1}^{4} \{ |x| + |3-x| \} dx$$

Sol.

$$I = \int_1^4 \{|x| + |3-x|\} dx$$

$$= \int_1^3 \{|x| + |3-x|\} dx + \int_3^4 \{|x| + |3-x|\} dx$$

$$= \int_1^3 3\, dx + \int_3^4 (2x-3) dx$$

$$= |3x|_1^3 + |x^2 - 3x|_3^4$$

$$= 6 + 4 = 10$$

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5.

Evaluate :

$$\int_{1}^{3} (|x-1| + |x-2|)\, dx$$

Sol.

$$I = \int_{1}^{3} (|x-1| + |x-2|)\, dx$$

$$= \int_{1}^{2} [(x-1)-(x-2)]\, dx + \int_{2}^{3} [(x-1)+(x-2)]\, dx$$

$$= \int_{1}^{2} 1\, dx + \int_{2}^{3} (2x-3)\, dx$$

$$= [x]_1^2 + [x^2 - 3x]_2^3$$

$$= 1 + 2 = 3$$

6.

Evaluate :

$$\int_{1}^{4} (|x-2| + |x-4|)\, dx$$

Sol.

$$\int_1^4 (|x-2| + |x-4|)\, dx$$

$$= \int_1^2 (2-x)\, dx + \int_2^4 (x-2) dx - \int_1^4 (x-4)\, dx$$

$$= \left[\frac{(2-x)^2}{-2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-4)^2}{2}\right]_1^4$$

$$= \frac{1}{2} + 2 + \frac{9}{2} = 7$$

7.

Evaluate :

$$\int_1^3 (|x-1|+|x-2|+|x-3|)\,dx$$

Sol.

$$\int_1^3 (|x-1| + |x-2| + |x-3|)dx$$

$$= \int_1^3 (x-1)dx + \int_1^2 -(x-2)dx + \int_2^3 (x-2)dx - \int_1^3 (x-3)dx$$

$$= \int_1^3 2\ dx + \int_1^2 (2-x)\ dx + \int_2^3 (x-2)dx$$

$$= [2x]_1^3 + \left[\frac{(2-x)^2}{-2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3$$

$$= 4 + \frac{1}{2} + \frac{1}{2} = 5$$

8.

Evaluate : $\int\limits_0^5 (|\ x-1\ | + |\ x-2\ | + |\ x-5\ |)\ dx$

Sol.

$$I = \int\limits_0^5 \left(|x-1| + |x-2| + |x-5|\right) dx$$

$$\therefore\ I = \left[-\int\limits_0^1 (x-1)dx + \int\limits_1^5 (x-1)dx\right] + \left[-\int\limits_0^2 (x-2)dx + \int\limits_2^5 (x-2)dx\right] + \left[-\int\limits_0^5 (x-5)dx\right]$$

$$= -\frac{(x-1)^2}{2}\Bigg]_0^1 + \frac{(x-1)^2}{2}\Bigg]_1^5 - \frac{(x-2)^2}{2}\Bigg]_0^2 + \frac{(x-2)^2}{2}\Bigg]_2^5 - \frac{(x-5)^2}{2}\Bigg]_0^5$$

$$= \frac{17}{2} + \frac{13}{2} + \frac{25}{2} = \frac{55}{2}$$

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VII. Property : Even/odd function function

1.

$$\text{Evaluate : } \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x \cdot \log\left(\frac{1+x}{1-x}\right) dx$$

Sol.

$$\text{Let } f(x) = \cos x . \log\left(\frac{1-x}{1+x}\right)$$

$$So,\ f(-x) = \cos(x).\log\left(\frac{1+x}{1-x}\right) = -f(x) \text{ [Odd function]}$$

$$\text{Thus, } I = \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x.\log\left(\frac{1-x}{1+x}\right) dx = 0$$

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2.

Evaluate :

$$\int_{-2}^{2} \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4} dx$$

Sol.

$I = \int_{-2}^{2} \frac{x^3+|x|+1}{x^2+4|x|+4} dx$

$= \int_{-2}^{2} \frac{x^3}{x^2+4|x|+4} dx + \int_{-2}^{2} \frac{|x|+1}{x^2+4|x|+4} dx$

$= I_1 + I_2 \ (say)$ --------(1)

$I_1 = 0 \ (\because \frac{x^3}{x^2+4|x|+4} \ is \ an \ odd \ function)$

$I_2 = 2 \int_0^2 \frac{x+1}{x^2+4x+4} dx \left(\because \frac{|x|+1}{x^2+4|x|+4} \ is \ an \ even \ function.\right)$

$= 2 \int_0^2 \frac{x+1}{(x+2)^2} dx$

Put x + 2 = t , so that dx = dt

$= 2 \int_2^4 \frac{t-1}{t^2} dt$

$= 2 \left[\int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) dt\right]$

$= 2 \left[\log|t| + \frac{1}{t}\right]_2^4$

$= 2 \left[\log 4 + \frac{1}{4} - \log 2 - \frac{1}{2}\right]$

$= 2 \log 2 - \frac{1}{2}$

Properties proof :

1.

f and g are continuous functions on interval [a, b]. Given that $f(a - x) = f(x)$

and $g(x) + g(a - x) = a$, show that $\int_0^a f(x)\, g(x)\, dx = \frac{a}{2} \int_0^a f(x)\, dx$.

Sol.

$$I = \int_0^a f(x)g(x)dx$$

$$= \int_0^a f(a-x)g(a-x)dx$$

$$= \int_0^a f(x)[a - g(x)]dx$$

$$I = a\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$$

Adding, we get $I = \frac{a}{2}\int_0^a f(x)dx$