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Board Booster – Perfect 100

041 Mathematics • CBSE Class XII

For CBSE Board 2026-27

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Master your Mother Book – NCERT before attempting this assignment.

Chapter 1 – Relations and Functions

1. Check whether the relation R defined on the set A = {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.

Solution : Write R in roster form:

R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

Reflexive: (1, 1) ∉ R, since 1 ≠ 1 + 1.

So R is not reflexive.

Symmetric: (1, 2) ∈ R but (2, 1) ∉ R.

So R is not symmetric.

Transitive: (1, 2) ∈ R and (2, 3) ∈ R, but (1, 3) ∉ R.

So R is not transitive.

∴ R is neither reflexive, nor symmetric, nor transitive.

2. Let A = {x ∈ Z : 0 ≤ x ≤ 12}. Show that R = {(a, b) : a, b ∈ A, |a − b| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

Solution : Reflexive: for any a ∈ A, |a − a| = 0, which is divisible by 4.

(a, a) ∈ R for all a ∈ A

Symmetric: let (a, b) ∈ R. Then |a − b| is divisible by 4.

|b − a| = |a − b| ⇒ |b − a| is divisible by 4 ⇒ (b, a) ∈ R

Transitive: let (a, b) ∈ R and (b, c) ∈ R. Then a − b = ±4m and b − c = ±4n.

a − c = (a − b) + (b − c) = ±4m ± 4n, which is divisible by 4 ⇒ (a, c) ∈ R

Hence R is an equivalence relation.

Elements related to 1 are those a ∈ A with |a − 1| divisible by 4:

{1, 5, 9}

Equivalence class of 2:

$[2] = \{2, 6, 10\}$

$\therefore$ R is an equivalence relation; elements related to 1 are $\{1, 5, 9\}$ and $[2] = \{2, 6, 10\}$.

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3. Show that the relation R in the set N × N defined by (a, b) R (c, d) if $a^2 + d^2 = b^2 + c^2$ for all a, b, c, d ∈ N is an equivalence relation.

Solution : Reflexive: $a^2 + b^2 = b^2 + a^2$, so (a, b) R (a, b) for every (a, b) ∈ N × N.

Symmetric: let (a, b) R (c, d).

$a^2 + d^2 = b^2 + c^2 \Rightarrow c^2 + b^2 = d^2 + a^2 \Rightarrow$ (c, d) R (a, b)

Transitive: let (a, b) R (c, d) and (c, d) R (e, f).

$a^2 + d^2 = b^2 + c^2$ …(1)

$c^2 + f^2 = d^2 + e^2$ …(2)

Adding (1) and (2):

$a^2 + d^2 + c^2 + f^2 = b^2 + c^2 + d^2 + e^2 \Rightarrow a^2 + f^2 = b^2 + e^2$

$\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is reflexive, symmetric and transitive; hence an equivalence relation.

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4. Show that the relation S defined on the set N × N by (a, b) S (c, d) if a + d = b + c is an equivalence relation.

Solution : Reflexive: a + b = b + a, so (a, b) S (a, b) for all (a, b) ∈ N × N.

Symmetric: let (a, b) S (c, d).

$a + d = b + c \Rightarrow c + b = d + a \Rightarrow$ (c, d) S (a, b)

Transitive: let (a, b) S (c, d) and (c, d) S (e, f).

$a + d = b + c$ …(1)

$c + f = d + e$ …(2)

Adding (1) and (2):

$a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow$ (a, b) S (e, f)

$\therefore$ S is an equivalence relation.

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5. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad(b + c) = bc(a + d). Show that R is an equivalence relation.

Solution : Dividing by abcd (all non-zero), the condition becomes

$$ad(b + c) = bc(a + d) \Leftrightarrow \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a} \Leftrightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

Reflexive: $\frac{1}{a} - \frac{1}{b} = \frac{1}{a} - \frac{1}{b}$, so (a, b) R (a, b).

Symmetric: if $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$, then $\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$, so (c, d) R (a, b).

Transitive: if (a, b) R (c, d) and (c, d) R (e, f), then

$\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}=\frac{1}{e}-\frac{1}{f} \Rightarrow (a, b)\ R\ (e, f)$

$\therefore$ R is an equivalence relation.

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6. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a − b} is an equivalence relation. Also find the equivalence class [0].

Solution : Reflexive: a − a = 0 is divisible by 2, so (a, a) ∈ R for all a ∈ Z.

Symmetric: if 2 divides a − b, then 2 divides −(a − b) = b − a.

$(a, b) \in R \Rightarrow (b, a) \in R$

Transitive: if a − b = 2m and b − c = 2n, then

$a - c = (a - b) + (b - c) = 2(m + n) \Rightarrow (a, c) \in R$

So R is an equivalence relation.

[0] consists of all integers a with a − 0 divisible by 2:

$[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$\therefore$ R is an equivalence relation and [0] is the set of all even integers.

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7. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2} is an equivalence relation. Consider three right-angled triangles T_1 with sides 3, 4, 5; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Solution : Reflexive: every triangle is similar to itself, so (T, T) ∈ R.

Symmetric: if T_1 is similar to T_2, then T_2 is similar to T_1.

Transitive: if $T_1 \sim T_2$ and $T_2 \sim T_3$, then $T_1 \sim T_3$.

So R is an equivalence relation.

Compare the ratios of corresponding sides:

T_1 and T_3: $\frac{3}{6}=\frac{4}{8}=\frac{5}{10}=\frac{1}{2} \Rightarrow T_1 \sim T_3$

T_1 and T_2: $\frac{3}{5} \neq \frac{4}{12} \Rightarrow$ not similar (similarly T_2, T_3 are not similar)

$\therefore$ R is an equivalence relation; only T_1 and T_3 are related.

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8. Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line y = 2x + 4.

Solution : Reflexive: every line is parallel to itself, so (L, L) ∈ R. (A line is taken to be parallel to itself.)

Symmetric: if $L_1 \parallel L_2$, then $L_2 \parallel L_1$.

Transitive: if $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then $L_1 \parallel L_3$.

So R is an equivalence relation.

Lines related to $y = 2x + 4$ are the lines parallel to it, i.e. with the same slope 2:

$y = 2x + c$, where $c \in R$

$\therefore$ The required set is $\{y = 2x + c : c \in R\}$.

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9. Let A = {1, 2, 3, 4, 5, 6, 7} and R = {(a, b) : both a and b are either odd or even}. Show that R is an equivalence relation. Hence find the equivalence class [1].

Solution : Reflexive: a and a are both odd or both even, so $(a, a) \in R$.

Symmetric: if a, b are both odd (or both even), so are b, a.

$(a, b) \in R \Rightarrow (b, a) \in R$

Transitive: if $(a, b) \in R$ and $(b, c) \in R$, then a, b, c all have the same parity as b.

$(a, c) \in R$

So R is an equivalence relation.

[1] is the set of elements of A having the same parity as 1 (odd):

$[1] = \{1, 3, 5, 7\}$

$\therefore$ R is an equivalence relation and $[1] = \{1, 3, 5, 7\}$.

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10. Given a non-empty set X, define the relation R in P(X) as follows: for $A, B \in P(X)$, $(A, B) \in R$ iff $A \subset B$. Prove that R is reflexive and transitive but not symmetric.

Solution : Reflexive: every set is a subset of itself, $A \subset A$, so $(A, A) \in R$.

Transitive: if $A \subset B$ and $B \subset C$, then $A \subset C$.

$(A, B) \in R, (B, C) \in R \Rightarrow (A, C) \in R$

Not symmetric: take any element $x \in X$, $A = \varphi$ and $B = \{x\}$.

$\varphi \subset \{x\}$ but $\{x\} \not\subset \varphi \Rightarrow (A, B) \in R$ but $(B, A) \notin R$

$\therefore$ R is reflexive and transitive but not symmetric.

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11. Let N be the set of all natural numbers and R be the relation in N defined by R = {(a, b) : a is a factor of b}. Show that R is reflexive and transitive but not symmetric.

Solution : Reflexive: every natural number is a factor of itself.

$(a, a) \in R$ for all $a \in N$

Transitive: if a is a factor of b and b is a factor of c, then $b = ka$ and $c = mb$.

$c = m(ka) = (mk)a \Rightarrow$ a is a factor of $c \Rightarrow (a, c) \in R$

Not symmetric:

2 is a factor of 4, so $(2, 4) \in R$; but 4 is not a factor of 2, so $(4, 2) \notin R$

$\therefore$ R is reflexive and transitive but not symmetric.

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12. Show that the relation R in the set of real numbers defined as $R = \{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution : Not reflexive: take $a = \frac{1}{2}$.

$\frac{1}{2} \le \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ is false $\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$

Not symmetric: $(1, 4) \in R$ since $1 \le 16$, but $4 \le 1^2$ is false.

$(4, 1) \notin R$

Not transitive: $(3, 2) \in R$ since $3 \le 4$, and $(2, 1.5) \in R$ since $2 \le 2.25$,

but $3 \le (1.5)^2 = 2.25$ is false $\Rightarrow (3, 1.5) \notin R$

$\therefore$ R is neither reflexive nor symmetric nor transitive.

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13. Let S be the set of all real numbers and R be the relation in S defined by R = {(a, b) : a ≤ b}. Show that R is reflexive and transitive but not symmetric.

Solution : Reflexive: $a \le a$ for every real a, so $(a, a) \in R$.

Transitive: $a \le b$ and $b \le c \Rightarrow a \le c$.

$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

Not symmetric:

$(1, 2) \in R$ since $1 \le 2$, but $2 \le 1$ is false, so $(2, 1) \notin R$

$\therefore$ R is reflexive and transitive but not symmetric.

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14. Determine whether the relation R defined on the set R of all real numbers as R = {(a, b) : a, b ∈ R and a − b + √3 ∈ S, where S is the set of all irrational numbers} is reflexive, symmetric and transitive.

Solution : Reflexive: $a - a + \sqrt{3} = \sqrt{3}$, which is irrational.

$(a, a) \in R$ for every real a, so R is reflexive.

Not symmetric: take $a = \sqrt{3}$, $b = 0$.

$a - b + \sqrt{3} = 2\sqrt{3} \in S$, so $(\sqrt{3}, 0) \in R$

$b - a + \sqrt{3} = 0 - \sqrt{3} + \sqrt{3} = 0 \notin S$, so $(0, \sqrt{3}) \notin R$

Not transitive: take $a = \sqrt{3}$, $b = 1$, $c = 2\sqrt{3}$.

$(\sqrt{3}, 1)$: $\sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3} - 1 \in S$

$(1, 2\sqrt{3})$: $1 - 2\sqrt{3} + \sqrt{3} = 1 - \sqrt{3} \in S$

$(\sqrt{3}, 2\sqrt{3})$: $\sqrt{3} - 2\sqrt{3} + \sqrt{3} = 0 \notin S$

$\therefore$ R is reflexive, but neither symmetric nor transitive.

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15. Show that the relation R on the set A = {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Solution : Symmetric: $(1, 2) \in R$ and $(2, 1) \in R$ — every pair has its reverse in R.

Not reflexive: $(1, 1) \notin R$.

Not transitive:

$(1, 2) \in R$ and $(2, 1) \in R$, but $(1, 1) \notin R$

$\therefore$ R is symmetric but neither reflexive nor transitive.

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16. Show that the function f : R → R defined by $f(x) = \frac{x}{x^2+1}$ is neither one-one nor onto.

Solution : Not one-one: take $x_1 = 2$ and $x_2 = \frac{1}{2}$.

$f(2) = \frac{2}{5}, f\left(\frac{1}{2}\right) = \frac{1/2}{1/4+1} = \frac{2}{5}$

$f(2) = f(1/2)$ but $2 \neq 1/2 \Rightarrow$ f is not one-one

Not onto: suppose $f(x) = 1$ for some real x.

$\frac{x}{x^2+1} = 1 \Rightarrow x^2 - x + 1 = 0$

Discriminant $= 1 - 4 = -3 < 0$, so no real x exists

So $1 \in R$ (co-domain) has no pre-image.

$\therefore$ f is neither one-one nor onto.

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17. Show that the function f : R → R given by $f(x) = 4x^3 + 7$ is a bijection.

Solution : One-one: let $f(x_1) = f(x_2)$.

$4x_1^3 + 7 = 4x_2^3 + 7 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

So f is one-one.

Onto: let $y \in R$. Put $f(x) = y$.

$4x^3 + 7 = y \Rightarrow x = \sqrt[3]{\frac{y-7}{4}} \in R$

$f(x) = 4 \cdot \frac{y-7}{4} + 7 = y$

Every y has a pre-image, so f is onto.

$\therefore$ f is a bijection.

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18. Show that f : N → N given by f(x) = x + 1 if x is odd, and f(x) = x − 1 if x is even, is one-one and onto.

Solution : One-one: let $f(x_1) = f(x_2)$.

Both odd: $x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$

Both even: $x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$

x_1 odd, x_2 even: $f(x_1) = x_1 + 1$ is even and $f(x_2) = x_2 - 1$ is odd, so they cannot be equal

So f is one-one.

Onto: let $y \in N$.

If y is odd, then $y + 1$ is even and $f(y + 1) = y$

If y is even, then $y - 1$ is odd and $f(y - 1) = y$

Every $y \in N$ has a pre-image.

$\therefore$ f is one-one and onto.

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19. Let A = R − {3} and B = R − {1}. Consider the function f : A → B defined by $f(x) = \frac{x-2}{x-3}$. Show that f is one-one and onto.

Solution : One-one: let $f(x_1) = f(x_2)$.

$$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$-3x_1 - 2x_2 = -3x_2 - 2x_1 \Rightarrow x_1 = x_2$

Onto: let $y \in B$ $(y \neq 1)$. Put $f(x) = y$.

$$\frac{x-2}{x-3} = y \Rightarrow x - 2 = xy - 3y \Rightarrow x = \frac{2-3y}{1-y}$$

x is defined since $y \neq 1$, and $x \neq 3$ ($x = 3$ would give $2 - 3y = 3 - 3y$, i.e. $2 = 3$, impossible). So $x \in A$ and $f(x) = y$.

$\therefore$ f is one-one and onto (bijective).

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20. Show that the function f : R → {x ∈ R : −1 < x < 1} defined by $f(x) = \frac{x}{1+|x|}$, x ∈ R is one-one and onto.

Solution : For $x \geq 0$, $f(x) = \frac{x}{1+x} \in [0, 1)$; for $x < 0$, $f(x) = \frac{x}{1-x} \in (-1, 0)$.

One-one: f(x) has the same sign as x, so $f(x_1) = f(x_2)$ forces x_1, x_2 to have the same sign.

Both ≥ 0: $\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 + x_1x_2 = x_2 + x_1x_2 \Rightarrow x_1 = x_2$

Both < 0: $\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 = x_2$

Onto: let $y \in (-1, 1)$.

$0 \leq y < 1$: take $x = \frac{y}{1-y} \geq 0$; then $f(x) = y$

$-1 < y < 0$: take $x = \frac{y}{1+y} < 0$; then $f(x) = y$

$\therefore$ f is one-one and onto.

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21. Show that the function f in $A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto its range. Hence find f^{-1}.

Solution : One-one: let $f(x_1) = f(x_2)$.

$(4x_1 + 3)(6x_2 - 4) = (4x_2 + 3)(6x_1 - 4)$

$-16x_1 + 18x_2 = -16x_2 + 18x_1 \Rightarrow 34x_2 = 34x_1 \Rightarrow x_1 = x_2$

Range and inverse: put $y = \frac{4x+3}{6x-4}$.

$6xy - 4y = 4x + 3 \Rightarrow x(6y - 4) = 4y + 3 \Rightarrow x = \frac{4y+3}{6y-4}$

This is defined for every $y \neq \frac{2}{3}$, so f is onto R − {2/3}.

∴ f is a bijection from A onto R − {2/3}, and $f^{-1}(y) = \frac{4y+3}{6y-4}$ (f is its own inverse).

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22. Prove that the function f : [0, ∞) → [−5, ∞) defined as $f(x) = 4x^2 + 4x - 5$ is both one-one and onto.

Solution : One-one: let $f(x_1) = f(x_2)$ with $x_1, x_2 \geq 0$.

$4(x_1^2 - x_2^2) + 4(x_1 - x_2) = 0 \Rightarrow 4(x_1 - x_2)(x_1 + x_2 + 1) = 0$

$x_1 + x_2 + 1 > 0$, so $x_1 = x_2$

Onto: let $y \geq -5$. Complete the square:

$y = 4x^2 + 4x - 5 = (2x + 1)^2 - 6 \Rightarrow 2x + 1 = \sqrt{y+6} \Rightarrow x = \frac{\sqrt{y+6} - 1}{2}$

Since $y \geq -5$, $y + 6 \geq 1$ and $\sqrt{y+6} \geq 1$, so $x \geq 0$ and $f(x) = y$.

∴ f is one-one and onto.

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23. Show that the function f : R → R defined by f(x) = 1 if x is rational and f(x) = −1 if x is irrational is many-one and not onto. Find (i) f(1/2) (ii) f(√2) (iii) f(2 + √3).

Solution : Many-one:

$f(1) = f(2) = 1$, but $1 \neq 2$

Not onto: the range of f is {−1, 1}.

$0 \in R$ (co-domain) has no pre-image

Values:

(i) 1/2 is rational ⇒ f(1/2) = 1

(ii) √2 is irrational ⇒ f(√2) = −1

(iii) 2 + √3 is irrational ⇒ f(2 + √3) = −1

∴ f is many-one and into; f(1/2) = 1, f(√2) = −1, f(2 + √3) = −1.

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24. Let f : R → R be defined by (i) f(x) = x + |x| (ii) $f(x) = x^2 + 1$. Determine whether or not f is onto.

Solution : (i) For $x \geq 0$, $f(x) = 2x \geq 0$; for $x < 0$, $f(x) = x - x = 0$.

So $f(x) \geq 0$ for every x; a negative number such as −1 has no pre-image

f is not onto

(ii) $x^2 + 1 \geq 1$ for every real x.

So 0 (or any $y < 1$) has no pre-image

f is not onto

∴ Neither function is onto.

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25. Let f : N → N be defined by f(n) = $\frac{n+1}{2}$ if n is odd and f(n) = $\frac{n}{2}$ if n is even. State whether f is bijective. Justify your answer.

Solution : Not one-one:

$f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$, but $1 \neq 2$

Onto: let $m \in N$. Then 2m is even and

$f(2m) = \frac{2m}{2} = m$

So every $m \in N$ has a pre-image.

∴ f is onto but not one-one; hence f is not bijective.

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26. A function f : R − {−1, 1} → R is defined by f(x) = $\frac{x}{x^2-1}$. (i) Check if f is one-one. (ii) Check if f is onto.

Solution : (i) One-one: take $x_1 = 2$ and $x_2 = -\frac{1}{2}$ in the domain:

$f(2) = \frac{2}{4-1} = \frac{2}{3}$; $f\left(-\frac{1}{2}\right) = \frac{-1/2}{\frac{1}{4}-1} = \frac{-1/2}{-3/4} = \frac{2}{3}$

$f(2) = f\left(-\frac{1}{2}\right)$ but $2 \neq -\frac{1}{2}$, so f is not one-one.

(ii) Onto: let $y \in R$ and solve $y = \frac{x}{x^2-1}$:

$y x^2 - x - y = 0$

If y = 0, then x = 0 ∈ domain.

If $y \neq 0$, the discriminant $1 + 4y^2 > 0$, so

$x = \frac{1 \pm \sqrt{1+4y^2}}{2y}$ is real

Also $x \neq \pm 1$, since $x = \pm 1$ gives $y \cdot 1 \mp 1 - y = \mp 1 \neq 0$. So every y has a pre-image in R − {−1, 1}.

∴ f is not one-one but it is onto.

Chapter 2 – Inverse Trigonometric Functions

1. Evaluate: $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$.

Solution : cos is an even function and has period 2π:

$$\cos\left(-\frac{7\pi}{3}\right)=\cos\frac{7\pi}{3}=\cos\left(2\pi+\frac{\pi}{3}\right)=\cos\frac{\pi}{3}$$

The principal value branch of $\cos^{-1}$ is $[0, \pi]$, and $\frac{\pi}{3}\in[0,\pi]$:

$$\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$$

$\therefore$ Value $=\frac{\pi}{3}$.

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2. Evaluate: $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$.

Solution : Reduce the angle using the period 2π:

$$\frac{14\pi}{3}=4\pi+\frac{2\pi}{3}\Rightarrow\cos\frac{14\pi}{3}=\cos\frac{2\pi}{3}$$

Since $\frac{2\pi}{3}\in[0,\pi]$ (principal branch of $\cos^{-1}$):

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$$

$\therefore$ Value $=\frac{2\pi}{3}$.

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3. Find the value of $\sin^{-1}\left(\cos\frac{43\pi}{5}\right)$.

Solution : Reduce the angle:

$$\frac{43\pi}{5}=8\pi+\frac{3\pi}{5}\Rightarrow\cos\frac{43\pi}{5}=\cos\frac{3\pi}{5}$$

Convert to sine so that the angle lies in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$:

$$\cos\frac{3\pi}{5}=\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\sin\left(-\frac{\pi}{10}\right)$$

$$\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right)=-\frac{\pi}{10}$$

$\therefore$ Value $=-\frac{\pi}{10}$.

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4. Evaluate: $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.

Solution : $\frac{3\pi}{5}$ does not lie in the principal branch $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so rewrite it:

$$\sin\frac{3\pi}{5}=\sin\left(\pi-\frac{3\pi}{5}\right)=\sin\frac{2\pi}{5}$$

$$\frac{2\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$\therefore \sin^{-1}\left(\sin\frac{3\pi}{5}\right)=\frac{2\pi}{5}$.

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5. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution : First term: $\frac{5\pi}{6}$ is not in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$$\tan\frac{5\pi}{6}=\tan\left(\pi-\frac{\pi}{6}\right)=\tan\left(-\frac{\pi}{6}\right)\Rightarrow\tan^{-1}\left(\tan\frac{5\pi}{6}\right)=-\frac{\pi}{6}$$

Second term:

$$\cos\frac{13\pi}{6}=\cos\left(2\pi+\frac{\pi}{6}\right)=\cos\frac{\pi}{6}\Rightarrow\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\frac{\pi}{6}$$

Sum:

$$-\frac{\pi}{6}+\frac{\pi}{6}=0$$

$\therefore$ Value $=0$.

---

6. Evaluate: $\cos\left\{\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$.

Solution : Principal value:

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\cos^{-1}\frac{\sqrt{3}}{2}=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$$

Substitute:

$$\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)=\cos\pi=-1$$

$\therefore$ Value $=-1$.

---

7. Evaluate: $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$.

Solution : Principal values:

$$\tan^{-1}\sqrt{3}=\frac{\pi}{3}\ (\text{since }\tan\frac{\pi}{3}=\sqrt{3})$$

$\sec^{-1}(-2) = \pi - \sec^{-1} 2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (range of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$)

Difference:

$$\frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

$\therefore$ Value $= -\frac{\pi}{3}$.

---

8. Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$.

Solution :

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$2\cos\left(2\times\frac{\pi}{6}\right) = 2\cos\frac{\pi}{3} = 2\times\frac{1}{2} = 1$$

$$\tan^{-1} 1 = \frac{\pi}{4}$$

$\therefore$ Value $= \frac{\pi}{4}$.

---

9. Find the value of $\cos\left(\tan^{-1}\left\{\cot\left(\sin^{-1}\frac{1}{2}\right)\right\}\right)$.

Solution : Work from the innermost function outwards:

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$\cot\frac{\pi}{6} = \sqrt{3}$$

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$\therefore$ Value $= \frac{1}{2}$.

---

10. Find the domain of $\sin^{-1}(x^2 - 4)$.

Solution : The domain of $\sin^{-1}$ is $[-1, 1]$, so

$$-1 \le x^2 - 4 \le 1 \Rightarrow 3 \le x^2 \le 5$$

$$\sqrt{3} \le |x| \le \sqrt{5}$$

$\therefore$ Domain $= [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$.

---

11. What is the domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$?

Solution : $\sqrt{x-1}$ is real only when $x \geq 1$, and $\sin^{-1}$ needs its argument in $[-1, 1]$:

$$0 \leq \sqrt{x-1} \leq 1 \Rightarrow 0 \leq x-1 \leq 1$$

$$1 \leq x \leq 2$$

$\therefore$ Domain = $[1, 2]$.

---

12. Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$.

Solution : Domain: $\sin^{-1} x$ needs $-1 \leq x \leq 1$, and $\sec^{-1} x$ needs $|x| \geq 1$.

Both hold only for $x = -1$ or $x = 1$

At $x = 1$:

$$f(1) = \frac{\pi}{2} + \frac{\pi}{4} + 0 = \frac{3\pi}{4}$$

At $x = -1$:

$$f(-1) = -\frac{\pi}{2} - \frac{\pi}{4} + \pi = \frac{\pi}{4}$$

$\therefore$ Range = $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

---

13. If $\sin^{-1} x + \sin^{-1} y = \pi$, find the value of $x^{2025} + y^{2025}$.

Solution : The greatest value of $\sin^{-1}$ is $\frac{\pi}{2}$, so the sum can equal π only if

$$\sin^{-1} x = \frac{\pi}{2} \text{ and } \sin^{-1} y = \frac{\pi}{2} \Rightarrow x = 1, y = 1$$

$$x^{2025} + y^{2025} = 1 + 1 = 2$$

$\therefore$ Value = 2.

---

14. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, find the value of $\cos^{-1} x + \cos^{-1} y$.

Solution : For $x \in [-1, 1]$, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, so $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$.

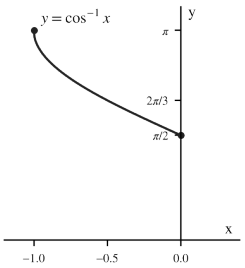
$$\cos^{-1} x + \cos^{-1} y = \pi - (\sin^{-1} x + \sin^{-1} y)$$

$$= \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

$\therefore \cos^{-1} x + \cos^{-1} y = \frac{\pi}{3}$.

---

15. Draw the graph of $\cos^{-1} x$, where $x \in [-1, 0]$. Also write its range.



Solution : On $[-1, 0]$, $y = \cos^{-1} x$ is decreasing. Key points:

$$x = -1: y = \pi; x = -\frac{\sqrt{3}}{2}: y = \frac{5\pi}{6}; x = -\frac{1}{2}: y = \frac{2\pi}{3}; x = 0: y = \frac{\pi}{2}$$

Plot these points and join them by a smooth curve (see figure).

$\therefore$ Range $= \left[\frac{\pi}{2}, \pi\right]$.

---

16. Write in the simplest form: $\tan^{-1} \sqrt{\frac{a - x}{a + x}}$, $-a < x < a$.

Solution : Put $x = a \cos\theta$, so $\theta = \cos^{-1} \frac{x}{a}$ and $\theta \in (0, \pi)$.

$$\frac{a - x}{a + x} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)} = \tan^2 \frac{\theta}{2}$$

Since $\frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$, $\tan\frac{\theta}{2} > 0$:

$$\tan^{-1} \sqrt{\tan^2 \frac{\theta}{2}} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore$ Simplest form $= \frac{1}{2}\cos^{-1} \frac{x}{a}$.

---

17. Write the simplest form of $\tan^{-1}\left[\frac{\sqrt{1 + x^2} - 1}{x}\right]$, $x \neq 0$.

Solution : Put $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\theta \neq 0$.

$$\sqrt{1 + x^2} = \sec\theta$$

$$\frac{\sec\theta - 1}{\tan\theta} = \frac{1 - \cos\theta}{\sin\theta} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan\frac{\theta}{2}$$

Since $\frac{\theta}{2} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$\tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore$ Simplest form $= \frac{1}{2}\tan^{-1} x$.

---

18. Find the domain of $f(x) = \sin^{-1}(2x - 1)$.

Solution : The argument of $\sin^{-1}$ must lie in $[-1, 1]$:

$-1 \le 2x - 1 \le 1 \Rightarrow 0 \le 2x \le 2 \Rightarrow 0 \le x \le 1$

$\therefore$ Domain = $[0, 1]$.

---

19. Find the domain of $f(x) = \cos^{-1}\left(\frac{x}{2} - 1\right)$.

Solution :

$-1 \le \frac{x}{2} - 1 \le 1 \Rightarrow 0 \le \frac{x}{2} \le 2 \Rightarrow 0 \le x \le 4$

$\therefore$ Domain = $[0, 4]$.

---

20. Find the domain of $f(x) = \sec^{-1}(2x + 1)$.

Solution : The domain of $\sec^{-1}$ is $R - (-1, 1)$, i.e. $|t| \ge 1$:

$|2x + 1| \ge 1 \Rightarrow 2x + 1 \le -1$ or $2x + 1 \ge 1$

$x \le -1$ or $x \ge 0$

$\therefore$ Domain = $(-\infty, -1] \cup [0, \infty)$.

---

21. Find the domain of $f(x) = \sin^{-1} x + \sin^{-1} 2x$.

Solution : Both terms must be defined:

$\sin^{-1} x$: $-1 \le x \le 1$

$\sin^{-1} 2x$: $-1 \le 2x \le 1 \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

Take the intersection:

$\therefore$ Domain = $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

---

22. Find the domain of $f(x) = \sin^{-1}(\log_2 x)$.

Solution : $\log_2 x$ is defined for $x > 0$, and $\sin^{-1}$ needs $-1 \le \log_2 x \le 1$:

$2^{-1} \le x \le 2^1$ ($\log_2$ is increasing)

$\therefore$ Domain = $\left[\frac{1}{2}, 2\right]$.

---

23. Find the range of $f(x) = 2\sin^{-1} x + \frac{\pi}{2}$.

Solution : The range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$-\pi \le 2\sin^{-1} x \le \pi$

$$-\frac{\pi}{2} \le 2\sin^{-1} x + \frac{\pi}{2} \le \frac{3\pi}{2}$$

$\therefore$ Range = $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

---

24. Find the range of $f(x) = 3\cos^{-1} x - \pi$.

Solution : The range of $\cos^{-1} x$ is $[0, \pi]$:

$$0 \le 3\cos^{-1} x \le 3\pi \Rightarrow -\pi \le 3\cos^{-1} x - \pi \le 2\pi$$

$\therefore$ Range = $[-\pi, 2\pi]$.

---

25. Find the domain and range of $f(x) = \sin^{-1}(x^2)$.

Solution : Domain: $-1 \le x^2 \le 1$ holds exactly when $-1 \le x \le 1$.

Domain = $[-1, 1]$

Range: for $x \in [-1, 1]$, x^2 takes all values in $[0, 1]$, and $\sin^{-1}$ is increasing:

$$\sin^{-1} 0 \le \sin^{-1}(x^2) \le \sin^{-1} 1$$

$\therefore$ Domain = $[-1, 1]$, Range = $\left[0, \frac{\pi}{2}\right]$.

---

26. Find the domain and range of $f(x) = \cos^{-1} |x|$.

Solution : Domain: $0 \le |x| \le 1 \Rightarrow -1 \le x \le 1$.

Range: $|x|$ takes all values in $[0, 1]$, and $\cos^{-1}$ is decreasing:

$$\cos^{-1} 1 \le \cos^{-1} |x| \le \cos^{-1} 0$$

$\therefore$ Domain = $[-1, 1]$, Range = $\left[0, \frac{\pi}{2}\right]$.

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27. Find the domain and range of $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$.

Solution : Domain: $\sin^{-1} x$ and $\cos^{-1} x$ need $x \in [-1, 1]$; $\tan^{-1} x$ is defined for all x.

Domain = $[-1, 1]$

For $x \in [-1, 1]$, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, so

$$f(x) = \frac{\pi}{2} + \tan^{-1} x$$

$\tan^{-1} x$ is increasing, so on $[-1, 1]$ it runs from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$:

$$\frac{\pi}{2} - \frac{\pi}{4} \le f(x) \le \frac{\pi}{2} + \frac{\pi}{4}$$

$\therefore$ Domain = $[-1, 1]$, Range = $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

---

28. Find the domain of $f(x) = \cos^{-1}(2x - 3) + \sin^{-1}\left(\frac{x}{2}\right)$.

Solution : Condition for $\cos^{-1}(2x - 3)$:

$-1 \le 2x - 3 \le 1 \Rightarrow 1 \le x \le 2$

Condition for $\sin^{-1}\frac{x}{2}$:

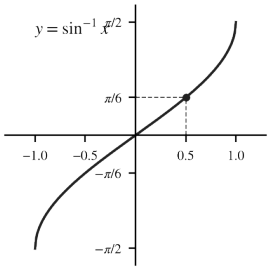
$-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

Both must hold, so take the intersection:

$\therefore$ Domain = [1, 2].

---

29. Draw the graph of $y = \sin^{-1} x$ and write its domain and range. On the same axes, mark the principal value $\sin^{-1}\left(\frac{1}{2}\right)$.



Solution : $y = \sin^{-1} x$ is the inverse of sine restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; it is increasing. Key points:

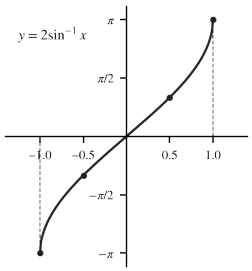
$(-1, -\frac{\pi}{2})$, $\left(-\frac{1}{2}, -\frac{\pi}{6}\right)$, $(0, 0)$, $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(1, \frac{\pi}{2}\right)$

Join these by a smooth curve (see figure). The point $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ gives $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

$\therefore$ Domain = $[-1, 1]$, Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

---

30. Draw the graph of the function $y = 2\sin^{-1} x$, $-\pi \le y \le \pi$. Also write its domain and range.



Solution : $y = 2\sin^{-1}x$ is defined when $-1 \le x \le 1$, and since $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$:

$-\pi \le 2\sin^{-1}x \le \pi$

Some points on the curve:

$x = -1$: $y = -\pi$; $x = -\frac{1}{2}$: $y = -\frac{\pi}{3}$; $x = 0$: $y = 0$; $x = \frac{1}{2}$: $y = \frac{\pi}{3}$; $x = 1$: $y = \pi$

Plot these points and join them by a smooth increasing curve (see figure).

$\therefore$ Domain = $[-1, 1]$; Range = $[-\pi, \pi]$.

---

31. Find the domain of the function $y = \cos^{-1}(|x - 1|)$. Show your steps.

Solution : The domain of $\cos^{-1}$ is $[-1, 1]$, so we need:

$-1 \le |x - 1| \le 1$

Since $|x - 1| \ge 0$ always, the left inequality holds for every x. The right one gives:

$|x - 1| \le 1 \Rightarrow -1 \le x - 1 \le 1 \Rightarrow 0 \le x \le 2$

$\therefore$ Domain = $[0, 2]$.

---

32. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, find the value of $\alpha(\beta + \gamma) - \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$.

Solution : The range of $\cos^{-1}$ is $[0, \pi]$, so each term is at most π. The sum 3π is possible only if:

$\cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi \Rightarrow \alpha = \beta = \gamma = -1$

$\alpha(\beta + \gamma) - \beta(\gamma + \alpha) + \gamma(\alpha + \beta) = (-1)(-2) - (-1)(-2) + (-1)(-2) = 2 - 2 + 2$

$\therefore$ Value = 2.

Chapter 3 – Matrices

1. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, show that $A^3 - 4A^2 - 3A + 11I = O$. Hence find A^{-1}.

Solution : Compute A^2 and A^3:

$$A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

Substitute:

$$A^3 - 4A^2 - 3A + 11I = \begin{bmatrix} 28 - 36 - 3 + 11 & 37 - 28 - 9 & 26 - 20 - 6 \\ 10 - 4 - 6 & 5 - 16 + 11 & 1 - 4 + 3 \\ 35 - 32 - 3 & 42 - 36 - 6 & 34 - 36 - 9 + 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}, multiply $A^3 - 4A^2 - 3A + 11I = O$ by A^{-1} ($|A| = -11 \ne 0$):

$$A^2 - 4A - 3I + 11A^{-1} = O \Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 4A - 3I)$$

$$A^2 - 4A - 3I = \begin{bmatrix} 9 - 4 - 3 & 7 - 12 & 5 - 8 \\ 1 - 8 & 4 - 3 & 1 + 4 \\ 8 - 4 & 9 - 8 & 9 - 12 - 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & -3 \\ -7 & 1 & 5 \\ 4 & 1 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & 3 \\ 7 & -1 & -5 \\ -4 & -1 & 6 \end{bmatrix}.$$

---

2. Find matrices X and Y if $2X + Y = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & 1 \end{bmatrix}$.

Solution : Add the two equations:

$$3X = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ -1 & -2 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{2}{3} & 0 & 2 \\ -\frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

From X − Y:

$$Y = X - \begin{bmatrix} 0 & 3 & 5 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -3 & -3 \\ \frac{5}{3} & \frac{10}{3} & \frac{1}{3} \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} \frac{2}{3} & 0 & 2 \\ -\frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \frac{2}{3} & -3 & -3 \\ \frac{5}{3} & \frac{10}{3} & \frac{1}{3} \end{bmatrix}.$$

---

3. Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$.

Solution : The left factor is 3 × 2 and the product is 3 × 2, so A is 2 × 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a - c & 2b - d \\ a & b \\ -3a + 4c & -3b + 4d \end{bmatrix}$$

Equate with the given matrix:

Row 2: a = 1, b = −2

Row 1: 2a − c = −1 ⇒ c = 3; 2b − d = −8 ⇒ d = 4

Row 3 (check): −3 + 12 = 9 ✓, 6 + 16 = 22 ✓

$$\therefore A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}.$$

---

4. If A is a square matrix such that $A^2 = I$, find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$.

Solution : Since $A^2 = I$, we get $A^3 = A^2A = A$. A commutes with I, so the binomial expansion holds:

$$(A - I)^3 = A^3 - 3A^2 + 3A - I = A - 3I + 3A - I = 4A - 4I$$

$$(A + I)^3 = A^3 + 3A^2 + 3A + I = A + 3I + 3A + I = 4A + 4I$$

Add and subtract 7A:

$(4A - 4I) + (4A + 4I) - 7A = A$

$\therefore$ The simplified value is A.

---

5. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$.

Solution :

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$A^2 - 5A + 4I = \begin{bmatrix} 5-10+4 & -1-0 & 2-5 \\ 9-10 & -2-5+4 & 5-15 \\ 0-5 & -1+5 & -2-0+4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

From $A^2 - 5A + 4I + X = O$:

$X = -(A^2 - 5A + 4I)$

$$\therefore A^2 - 5A + 4I = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix} \text{ and } X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}.$$

---

6. Express the matrix $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution :

$$A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Symmetric part:

$$P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix}$$

$P' = P$, so P is symmetric

Skew-symmetric part:

$$Q = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

$Q' = -Q$, so Q is skew-symmetric

$\therefore$ A = P + Q with P and Q as above.

---

7. Show that the matrix B′AB is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Solution : Use $(XYZ)' = Z'Y'X'$ and $(B')' = B$:

$(B'AB)' = B'A'(B')' = B'A'B$

Case 1: A is symmetric ($A' = A$):

$(B'AB)' = B'AB \Rightarrow B'AB$ is symmetric

Case 2: A is skew-symmetric ($A' = -A$):

$(B'AB)' = B'(-A)B = -(B'AB) \Rightarrow B'AB$ is skew-symmetric

$\therefore$ B′AB is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

---

8. If A and B are symmetric matrices of the same order, show that AB − BA is a skew-symmetric matrix.

Solution : Given $A' = A$ and $B' = B$.

$(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$

$\therefore$ AB − BA is skew-symmetric.

---

9. For what value of x is the matrix $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ skew-symmetric?

Solution : For a skew-symmetric matrix, $a_{ij} = -a_{ji}$.

$a_{31} = -a_{13} \Rightarrow x = -(-2) = 2$

The other entries already satisfy the condition (diagonal entries 0; $a_{12} = -a_{21}$, $a_{23} = -a_{32}$).

$\therefore x = 2$.

10. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric. Find the values of a and b.

Solution : For a symmetric matrix, $A' = A$, i.e. $a_{ij} = a_{ji}$.

$a_{12} = a_{21} \Rightarrow 2b = 3 \Rightarrow b = \frac{3}{2}$

$a_{13} = a_{31} \Rightarrow -2 = 3a \Rightarrow a = -\frac{2}{3}$

$\therefore a = -\frac{2}{3}, b = \frac{3}{2}$.

11. Find x, y, z if $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A' = A^{-1}$.

Solution : $A' = A^{-1}$ means $A'A = I$.

$$A'A = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix}$$

Equate with I:

$2x^2 = 1, 6y^2 = 1, 3z^2 = 1$

$\therefore x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$.

12. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$, find the value of (ad − bc), given that ad − bc < 0.

Solution : For 2×2 matrices, $|A^2| = |A|^2$, and $|A| = ad - bc$.

$|A^2| = \begin{vmatrix} 7 & 12 \\ 18 & 31 \end{vmatrix} = 217 - 216 = 1$

$(ad - bc)^2 = 1 \Rightarrow ad - bc = \pm 1$

Since $ad - bc < 0$:

$\therefore ad - bc = -1$.

13. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and I is the identity matrix of order 2, show that $A^2 = 4A - 3I$. Hence find A^{-1}.

Solution :

$$A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$4A - 3I = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

So $A^2 = 4A - 3I$. Multiply by A^{-1} ($|A| = 3 \neq 0$):

$$A = 4I - 3A^{-1} \Rightarrow A^{-1} = \frac{1}{3}(4I - A)$$

$$4I - A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

---

14. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

Solution :

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiply $A^2 - 5A + 7I = O$ by A^{-1} ($|A| = 7 \neq 0$):

$$A - 5I + 7A^{-1} = O \Rightarrow A^{-1} = \frac{1}{7}(5I - A)$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

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15. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = O$.

Solution :

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Substitute:

$$A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 - 30 + 7 + 2 & 0 & 34 - 48 + 14 \\ 12 - 12 + 0 & 8 - 24 + 14 + 2 & 23 - 30 + 7 \\ 34 - 48 + 14 & 0 & 55 - 78 + 21 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 7A + 2I = O.$

---

16. If A is a square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.

Solution : From $A^2 = A$: $A^3 = A^2A = A \cdot A = A^2 = A$. Since A and I commute:

$(I + A)^3 = I^3 + 3I^2A + 3IA^2 + A^3$

$= I + 3A + 3A + A = I + 7A$

$\therefore (I + A)^3 = 7A + I.$

---

17. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, find α $(0 < \alpha < \frac{\pi}{2})$ satisfying $A + A' = I$.

Solution :

$$A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A + A' = \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix}$$

Equate with I:

$$2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2}$$

$\therefore \alpha = \frac{\pi}{3}.$

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18. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that $I + A = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Solution : Let $t = \tan\frac{\alpha}{2}$. Then

$$\cos\alpha = \frac{1 - t^2}{1 + t^2}, \sin\alpha = \frac{2t}{1 + t^2}$$

$$I + A = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}, I - A = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$$

RHS:

$$\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha + t\sin\alpha & -\sin\alpha + t\cos\alpha \\ -t\cos\alpha + \sin\alpha & t\sin\alpha + \cos\alpha \end{bmatrix}$$

Simplify each entry:

$$\cos\alpha + t\sin\alpha = \frac{1 - t^2 + 2t^2}{1 + t^2} = 1$$

$$-\sin\alpha + t\cos\alpha = \frac{-2t + t - t^3}{1 + t^2} = \frac{-t(1 + t^2)}{1 + t^2} = -t$$

$$\sin\alpha - t\cos\alpha = t$$

So RHS $= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = I + A.$

$$\therefore I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

---

19. Find the inverse of the matrix $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$. Hence find the matrix P satisfying the matrix equation $P\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

$|A| = 9 - 10 = -1 \neq 0$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = -1 \times \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

From $PA = B$, multiply on the right by A^{-1}:

$$P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 + 10 & 2 + 6 \\ 6 - 5 & 4 - 3 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 13 & 8 \\ 1 & 1 \end{bmatrix}.$$

---

20. Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

Solution : X must be 2×2. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a + 4b & 2a + 5b & 3a + 6b \\ c + 4d & 2c + 5d & 3c + 6d \end{bmatrix}$$

First row:

$a + 4b = -7$, $2a + 5b = -8 \Rightarrow b = -2$, $a = 1$ (check: $3 - 12 = -9$ ✓)

Second row:

$c + 4d = 2$, $2c + 5d = 4 \Rightarrow d = 0, c = 2$ (check: $6 + 0 = 6$ ✓)

$\therefore X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

---

21. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find the values of a and b.

Solution : $(A + B)^2 = A^2 + AB + BA + B^2$, so the condition gives $AB + BA = O$.

$$AB = \begin{bmatrix} a - b & 2 \\ 2a - b & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} a + 2 & -a - 1 \\ b - 2 & -b + 1 \end{bmatrix}$$

$$AB + BA = \begin{bmatrix} 2a - b + 2 & 1 - a \\ 2a - 2 & 4 - b \end{bmatrix}$$

Equate to O:

$1 - a = 0 \Rightarrow a = 1$; $4 - b = 0 \Rightarrow b = 4$

Check: $2a - b + 2 = 2 - 4 + 2 = 0$ ✓, $2a - 2 = 0$ ✓

$\therefore a = 1, b = 4$.

---

22. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds so as to obtain an annual total interest of ₹ 1800.

Solution : Let ₹ x be invested in the first bond; then ₹ $(30000 - x)$ is in the second.

Annual interest by matrix multiplication:

$$\begin{bmatrix} x & 30000 - x \end{bmatrix} \begin{bmatrix} 0.05 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 1800 \end{bmatrix}$$

$0.05x + 0.07(30000 - x) = 1800$

$0.05x + 2100 - 0.07x = 1800$

$-0.02x = -300 \Rightarrow x = 15000$

$\therefore$ ₹ 15,000 in each type of bond.

---

23. Theorem: Prove that all the diagonal elements of a skew-symmetric matrix are zero.

Solution : Let $A = [a_{ij}]$ be a skew-symmetric matrix of order n. Then $A' = -A$, so for all i, j:

$a_{ji} = -a_{ij}$

Put $j = i$ (a diagonal element):

$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$ for $i = 1, 2, \ldots, n$

$\therefore$ Every diagonal element of a skew-symmetric matrix is zero.

---

24. Theorem: Prove that the determinant of a skew-symmetric matrix of odd order is zero. Is the result true for even order?

Solution : Let A be skew-symmetric of odd order n, so $A' = -A$. Take determinants:

$|A'| = |-A|$

Use $|A'| = |A|$ and $|kA| = k^n|A|$ with $k = -1$:

$|A| = (-1)^n|A| = -|A|$ (n is odd)

$2|A| = 0 \Rightarrow |A| = 0$

Even order: the result need not hold. For example,

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric (order 2), but $|A| = 0 - (-1) = 1 \neq 0$

$\therefore |A| = 0$ for every skew-symmetric matrix of odd order; for even order it may be non-zero.

---

25. If A is a square matrix which is both symmetric and skew-symmetric, show that A is a zero matrix.

Solution : Symmetric: $A' = A$. Skew-symmetric: $A' = -A$.

$A = -A \Rightarrow 2A = O \Rightarrow A = O$

$\therefore$ A is the zero matrix.

---

26. For any square matrix A, show that $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

Solution : Use $(A')' = A$ and $(B \pm C)' = B' \pm C'$.

$(A + A')' = A' + (A')' = A' + A = A + A' \Rightarrow A + A'$ is symmetric

$(A - A')' = A' - (A')' = A' - A = -(A - A') \Rightarrow A - A'$ is skew-symmetric

$\therefore A + A'$ is symmetric and $A - A'$ is skew-symmetric.

---

27. If A and B are symmetric matrices of the same order, prove that AB is symmetric if and only if AB = BA.

Solution : Given $A' = A$ and $B' = B$, so $(AB)' = B'A' = BA$.

If AB is symmetric:

$(AB)' = AB \Rightarrow BA = AB$

Conversely, if $AB = BA$:

$(AB)' = BA = AB \Rightarrow AB$ is symmetric

$\therefore$ AB is symmetric if and only if A and B commute.

---

28. If A is a matrix of order 3×4 and B is a matrix such that both $A'B$ and BA' are defined, find the order of B.

Solution : A' is of order 4×3. Let B be of order $m \times n$.

$A'B$ is defined $\Rightarrow$ number of columns of A' = number of rows of $B \Rightarrow m = 3$

BA' is defined $\Rightarrow$ number of columns of B = number of rows of $A' \Rightarrow n = 4$

$\therefore$ B is of order 3×4.

---

29. Find the number of all possible matrices of order 3 × 3 with each entry 0 or 1. How many of them are symmetric?

Solution : A 3 × 3 matrix has 9 entries, each with 2 choices:

Total = $2^9 = 512$

A symmetric matrix is fixed by its 3 diagonal entries and the 3 entries above the diagonal (6 free entries):

Symmetric ones = $2^6 = 64$

∴ 512 matrices in all, of which 64 are symmetric.

---

30. Give an example of two non-zero matrices A and B such that AB = O. What does this show?

Solution : Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, both non-zero.

$$AB = \begin{bmatrix} 0\cdot 1 + 1\cdot 0 & 0\cdot 0 + 1\cdot 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

∴ AB = O although A ≠ O and B ≠ O; so for matrices, AB = O does not imply A = O or B = O.

Chapter 4 – Determinants

1. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

Solution :

$|A| = 14 - 12 = 2 \neq 0$, so A^{-1} exists

adj $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ (swap the diagonal entries, change the signs of the others)

$$A^{-1} = \frac{1}{|A|}\,\text{adj } A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Now

$$2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ and $2A^{-1} = 9I - A$.

---

2. If A is a square matrix satisfying A′A = I, write the possible values of |A|.

Solution : Take determinants on both sides:

$|A'A| = |I| \Rightarrow |A'|\,|A| = 1$

$|A'| = |A| \Rightarrow |A|^2 = 1$

∴ $|A| = 1$ or $|A| = -1$.

3. If A is a 3 × 3 matrix, |A| ≠ 0 and |3A| = k|A|, write the value of k. Also, if |A| = 5, find |2A′|.

Solution : For a square matrix of order n, $|kA| = k^n|A|$.

$|3A| = 3^3|A| = 27|A| \Rightarrow k = 27$

For |A| = 5:

$|2A'| = 2^3|A'| = 8|A| = 8 \times 5 = 40$

∴ k = 27 and |2A′| = 40.

4. If A is a non-singular matrix of order 3 with |A| = −4, find |A · adj A|. If A and B are square matrices of order 3 with |A| = 2 and AB = 2I, find |B|.

Solution : Part 1: A(adj A) = |A| I, so

$|A \cdot \text{adj } A| = |\,|A|\, I\,| = |A|^3|I| = (-4)^3 = -64$

Part 2: take determinants of AB = 2I:

$|A|\,|B| = |2I| = 2^3 = 8$

$2|B| = 8 \Rightarrow |B| = 4$

∴ |A · adj A| = −64 and |B| = 4.

5. If A and B are invertible square matrices of the same order, prove that adj(AB) = (adj B)(adj A).

Solution : For any invertible matrix X, adj X = $|X|\,X^{-1}$. Also $(AB)^{-1} = B^{-1}A^{-1}$ and |AB| = |A||B|.

$\text{adj}(AB) = |AB|\,(AB)^{-1} = |A|\,|B|\,B^{-1}A^{-1}$

$(\text{adj } B)(\text{adj } A) = (|B|\,B^{-1})(|A|\,A^{-1}) = |A|\,|B|\,B^{-1}A^{-1}$

∴ adj(AB) = (adj B)(adj A).

6. Find the equation of the line joining A(1, 3) and B(0, 0) using determinants, and find the value of k if D(k, 0) is a point such that the area of ΔABD is 3 square units.

Solution : Let P(x, y) be any point on AB. Then A, B, P are collinear, so the area of ΔABP is 0:

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

Expand along the second row:

$-1 \times (y - 3x) = 0 \Rightarrow y = 3x$

Area of ΔABD:

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

$$\frac{1}{2}[-1 \times (0 - 3k)] = \frac{3k}{2} = \pm 3 \Rightarrow k = \pm 2$$

∴ Equation of AB: y = 3x; k = ±2.

7. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

Solution :

$|A| = 1(-1 - 8) + 2(0 + 8) + 3(0 - 2) = -9 + 16 - 6 = 1$

Cofactors of A:

$A_{11} = -9, A_{12} = -8, A_{13} = -2; A_{21} = 8, A_{22} = 7, A_{23} = 2; A_{31} = -5, A_{32} = -4, A_{33} = -1$

$\text{adj } A = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A^{-1} = \text{adj } A \text{ (since } |A| = 1)$

Use $(A')^{-1} = (A^{-1})'$:

$\therefore (A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}.$

---

8. Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that $A(\text{adj } A) = |A| I_3$.

Solution : Cofactors:

$A_{11} = 1 - 4 = -3, A_{12} = -(2 + 4) = -6, A_{13} = -4 - 2 = -6$

$A_{21} = -(-2 - 4) = 6, A_{22} = -1 + 4 = 3, A_{23} = -(2 + 4) = -6$

$A_{31} = 4 + 2 = 6, A_{32} = -(2 + 4) = -6, A_{33} = -1 + 4 = 3$

$\text{adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

$|A| = -1(1 - 4) + 2(2 + 4) - 2(-4 - 2) = 3 + 12 + 12 = 27$

Multiply:

$A(\text{adj } A) = \begin{bmatrix} 3 + 12 + 12 & -6 - 6 + 12 & -6 + 12 - 6 \\ -6 - 6 + 12 & 12 + 3 + 12 & 12 - 6 - 6 \\ -6 + 12 - 6 & 12 - 6 - 6 & 12 + 12 + 3 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$

$\therefore A(\text{adj } A) = 27 I_3 = |A| I_3.$

---

9. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, find the values of λ for which A^{-1} exists.

Solution : A^{-1} exists if and only if $|A| \neq 0$.

$|A| = 2(6 - 5) - \lambda(0 - 5) + (-3)(0 - 2) = 2 + 5\lambda + 6 = 8 + 5\lambda$

$8 + 5\lambda \neq 0 \Rightarrow \lambda \neq -\frac{8}{5}$

$\therefore A^{-1}$ exists for all real λ except $\lambda = -\frac{8}{5}$.

10. If $x \in R$ and $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, find the value of x.

Solution : Expand both determinants:

$(x + 1)(x + 2) - (x - 1)(x - 3) = 12 + 1$

$(x^2 + 3x + 2) - (x^2 - 4x + 3) = 13$

$7x - 1 = 13$

$\therefore x = 2.$

11. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

Solution :

$|B| = 1(3 - 0) - 2(-1 - 0) - 2(2 - 0) = 3 + 2 - 4 = 1$

Cofactors of B give

$$\text{adj } B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \Rightarrow B^{-1} = \text{adj } B$$

Use $(AB)^{-1} = B^{-1}A^{-1}$:

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

12. If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$, find A^{-1} and hence solve the system: $3x + 4y + 7z = 14$, $2x - y + 3z = 4$, $x + 2y - 3z = 0$.

Solution :

$|A| = 3(3 - 6) - 2(-12 - 14) + 1(12 + 7) = -9 + 52 + 19 = 62$

$$\text{adj } A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}$$

The coefficient matrix of the system is A′, so $X = (A')^{-1}B = (A^{-1})'B$:

$$X = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -42+104 \\ 126-64 \\ 70-8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x = 1, y = 1, z = 1.$

---

13. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$, find A^{-1} and use it to solve the system: $x + 2y - 3z = 6$, $3x + 2y - 2z = 3$, $2x - y + z = 2$.

Solution :

$|A| = 1(2-2) - 2(3+4) - 3(-3-4) = 0 - 14 + 21 = 7$

$$\text{adj } A = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$$

The system is $AX = B$ with $B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$:

$$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 0+3+4 \\ -42+21-14 \\ -42+15-8 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix}$$

$\therefore x = 1, y = -5, z = -5.$

---

14. Use the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system: $x + 3z = 9$, $-x + 2y - 2z = 4$, $2x - 3y + 4z = -3$.

Solution : Let $P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$.

$$PQ = \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = I \Rightarrow P^{-1} = Q$$

The coefficient matrix of the system is $C = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix} = P'$, so

$C^{-1} = (P')^{-1} = (P^{-1})' = Q'$

$$X = Q'B = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\therefore x = 0, y = 5, z = 3.$

---

15. If $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$, find A^{-1}. Hence solve the system: $x + 2y + z = 4$, $-x + y + z = 0$, $x - 3y + z = 4$.

Solution :

$|A| = 1(1+3) - 2(-1-1) + 1(3-1) = 4 + 4 + 2 = 10$

$$\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

The system is $AX = B$ with $B = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$:

$$X = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ 8+0-8 \\ 8+0+12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix}$$

$\therefore x = 2, y = 0, z = 2.$

---

16. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, find AB and hence solve the system: $x - y = 3$, $2x + 3y + 4z = 17$, $y + 2z = 7$.

Solution :

$$AB = \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

So $A^{-1} = \frac{1}{6}B$. The system is $AX = C$ with $C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$:

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$$X = \frac{1}{6}BC = \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix}$$

$\therefore x = 2, y = -1, z = 4.$

---

17. Solve the system of equations: $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4, \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1, \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2.$

Solution : Put $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$: the system becomes AX = B with

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$|A| = 2(120-45) - 3(-80-30) + 10(36+36) = 150 + 330 + 720 = 1200$

$$\text{adj } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

So $u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5}$.

$\therefore x = 2, y = 3, z = 5.$

---

18. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find the cost of each item per kg by the matrix method.

Solution : Let the cost per kg of onion, wheat and rice be ₹ x, ₹ y, ₹ z.

$4x + 3y + 2z = 60, 2x + 4y + 6z = 90, 6x + 2y + 3z = 70$

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$|A| = 4(12-12) - 3(6-36) + 2(4-24) = 0 + 90 - 40 = 50$

$$\text{adj } A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$\therefore$ Onion ₹ 5/kg, wheat ₹ 8/kg, rice ₹ 8/kg.

---

19. For the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, show that $A^3 - 6A^2 + 9A - 4I = O$. Hence find A^{-1}.

Solution :

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}, A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix} = O$$

Multiply by A^{-1} ($|A| = 4 \neq 0$):

$$A^2 - 6A + 9I - 4A^{-1} = O \Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$A^2 - 6A + 9I = \begin{bmatrix} 6-12+9 & -5+6 & 5-6 \\ -5+6 & 6-12+9 & -5+6 \\ 5-6 & -5+6 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

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20. If $A = [a_{ij}]$ is a matrix of order 2×2 such that $|A| = -15$ and A_{ij} is the cofactor of a_{ij}, find $a_{21}A_{21} + a_{22}A_{22}$.

Solution : The sum of the products of the elements of any row with their own cofactors equals the determinant.

$a_{21}A_{21} + a_{22}A_{22}$ = expansion of $|A|$ along the second row

$\therefore a_{21}A_{21} + a_{22}A_{22} = |A| = -15.$

---

21. If A is a square matrix of order 3 with $|A| = 5$, find $|\text{adj } A|$ and $|A^{-1}|$.

Solution : For a square matrix of order n, $|\text{adj } A| = |A|^{n-1}$ and $|A^{-1}| = \frac{1}{|A|}$.

$|\text{adj } A| = 5^{3-1} = 25$

$|A^{-1}| = \frac{1}{5}$

$\therefore |\text{adj } A| = 25$ and $|A^{-1}| = \frac{1}{5}$.

---

22. If A is a square matrix of order 3 and |A| = 3, find |adj(adj A)|.

Solution : Apply $|\text{adj } B| = |B|^{n-1}$ twice (n = 3):

$|\text{adj } A| = |A|^2 = 9$

$|\text{adj}(\text{adj } A)| = |\text{adj } A|^2 = 9^2 = 81$

(In general $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2} = 3^4$.)

$\therefore |\text{adj}(\text{adj } A)| = 81.$

---

23. If A is an invertible matrix of order 3 with |A| = 2, find $|3A^{-1}|$.

Solution :

$$|3A^{-1}| = 3^3|A^{-1}| = 27 \times \frac{1}{|A|}$$

$$= 27 \times \frac{1}{2}$$

$$\therefore |3A^{-1}| = \frac{27}{2}.$$

---

24. If A is a square matrix of order 3 with |A| = −2, find |−2A| and |AA′|.

Solution :

$|-2A| = (-2)^3|A| = -8 \times (-2) = 16$

$|AA'| = |A|\,|A'| = |A| \cdot |A| = (-2)^2 = 4$

$\therefore |-2A| = 16$ and $|AA'| = 4.$

---

25. If A is a square matrix such that $A^2 = A$, show that |A| = 0 or |A| = 1.

Solution : Take determinants of $A^2 = A$:

$|A|^2 = |A| \Rightarrow |A|(|A| - 1) = 0$

$\therefore |A| = 0$ or $|A| = 1.$

---

26. If x ∈ N and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, find the value of x.

Solution : Expand the determinant:

$2x(x + 3) - (-2)(-3x) = 8$

$2x^2 + 6x - 6x = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Since $x \in N$:

$\therefore x = 2.$

---

27. For what value of x is the matrix $\begin{bmatrix} 2(x+1) & 2x \\ x & x-2 \end{bmatrix}$ singular?

Solution : A singular matrix has determinant zero:

$2(x + 1)(x - 2) - 2x \cdot x = 0$

$2x^2 - 2x - 4 - 2x^2 = 0 \Rightarrow -2x - 4 = 0$

$\therefore x = -2.$

---

28. Using determinants, find the value of k if the area of the triangle with vertices (k, 0), (4, 0) and (0, 2) is 4 square units.

Solution :

$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}\right| = 4$$

Expand along the first row:

$k(0 - 2) - 0 + 1(8 - 0) = 8 - 2k$

$\frac{1}{2}|8 - 2k| = 4 \Rightarrow |8 - 2k| = 8 \Rightarrow 8 - 2k = \pm 8$

$\therefore k = 0$ or $k = 8$.

---

29. If A is a 3 × 3 invertible matrix, show that for any scalar $k \neq 0$, $(kA)^{-1} = \frac{1}{k}A^{-1}$. Hence calculate $(3A)^{-1}$, where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution :

$$(kA)\left(\frac{1}{k}A^{-1}\right) = k \cdot \frac{1}{k}(AA^{-1}) = I, \text{ and similarly } \left(\frac{1}{k}A^{-1}\right)(kA) = I$$

So $\frac{1}{k}A^{-1}$ is the inverse of kA.

For the given A:

$|A| = 2(4 - 1) + 1(-2 + 1) + 1(1 - 2) = 6 - 1 - 1 = 4$

$$\text{adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{4}\text{adj } A$$

$$\therefore (3A)^{-1} = \frac{1}{3}A^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

---

30. If $A = \begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix}$, find the value of (a + x) − (b + y).

Solution : Since $AA^{-1} = I$:

Row 1 × Col 1: $-1 - 8a + 2b = 1$; Row 1 × Col 2: $1 + 7a + 2y = 0$; Row 1 × Col 3: $-1 - 5a + 6 = 0$

From the third: $a = 1$. Then $2b = 2 + 8 \Rightarrow b = 5$ and $2y = -8 \Rightarrow y = -4$

Row 2 × Col 3: $1 - 10 + 3x = 0 \Rightarrow x = 3$

$\therefore (a + x) - (b + y) = 4 - 1 = 3.$

Note: Case-study questions are recommended from this chapter for **CBSE Board 2026-27**.
Practise them here: **Class XII Case Studies (click here)**

Chapter 5 – Continuity and Differentiability

1. Find the value(s) of λ if the function $f(x) = \dfrac{\sin^2 \lambda x}{x^2}$ for $x \neq 0$, and $f(x) = 1$ for $x = 0$, is continuous at $x = 0$.

Solution : For continuity at 0, $\lim\limits_{x \to 0} f(x) = f(0) = 1$.

$$\lim_{x \to 0} \frac{\sin^2 \lambda x}{x^2} = \lim_{x \to 0} \left(\frac{\sin \lambda x}{\lambda x}\right)^2 \lambda^2 = \lambda^2$$

$$\lambda^2 = 1$$

$\therefore \lambda = \pm 1.$

---

2. Find the values of p and q for which $f(x) = \dfrac{1 - \sin^3 x}{3\cos^2 x}$ if $x < \dfrac{\pi}{2}$; $f(x) = p$ if $x = \dfrac{\pi}{2}$; $f(x) = \dfrac{q(1 - \sin x)}{(\pi - 2x)^2}$ if $x > \dfrac{\pi}{2}$, is continuous at $x = \dfrac{\pi}{2}$.

Solution : Left-hand limit: let $s = \sin x \to 1$. Factorise $1 - s^3 = (1 - s)(1 + s + s^2)$ and $\cos^2 x = (1 - s)(1 + s)$:

$$\text{LHL} = \lim \frac{(1 - s)(1 + s + s^2)}{3(1 - s)(1 + s)} = \frac{1 + 1 + 1}{3 \times 2} = \frac{1}{2}$$

Right-hand limit: put $x = \dfrac{\pi}{2} + h$, $h \to 0^+$. Then $\sin x = \cos h$ and $\pi - 2x = -2h$:

$$\text{RHL} = \lim_{h \to 0} \frac{q(1 - \cos h)}{4h^2} = \lim_{h \to 0} \frac{q \cdot 2\sin^2 (h/2)}{4h^2} = \frac{q}{8}$$

For continuity, LHL = RHL = $f(\pi/2)$:

$$\frac{1}{2} = \frac{q}{8} = p$$

$\therefore p = \dfrac{1}{2}, q = 4.$

---

3. Find the value of k for which f(x) = $\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$ if $-1 \le x < 0$, and f(x) = $\frac{2x+1}{x-1}$ if $0 \le x < 1$, is continuous at x = 0.

Solution :

$f(0) = \frac{0+1}{0-1} = -1$, and RHL = −1

Left-hand limit (rationalise):

$$\lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x(\sqrt{1+kx}+\sqrt{1-kx})} = \lim_{x\to 0^-} \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}} = \frac{2k}{2} = k$$

For continuity:

$k = -1$

$\therefore k = -1$.

---

4. If f(x) = $\frac{1-\cos 4x}{x^2}$ when x < 0; f(x) = a when x = 0; f(x) = $\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$ when x > 0, and f is continuous at x = 0, find the value of a.

Solution : Left-hand limit:

$$\lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2} = \lim_{x\to 0^-} 8\left(\frac{\sin 2x}{2x}\right)^2 = 8$$

Right-hand limit (rationalise the denominator):

$$\lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{(16+\sqrt{x})-16} = \lim_{x\to 0^+} (\sqrt{16+\sqrt{x}}+4) = 8$$

For continuity, f(0) = 8:

$\therefore a = 8$.

---

5. Find the relationship between a and b so that the function f(x) = ax + 1 if $x \le 3$, and f(x) = bx + 3 if x > 3, is continuous at x = 3.

Solution :

f(3) = 3a + 1 = LHL

RHL = $\lim_{x\to 3^+} (bx+3) = 3b + 3$

For continuity:

$3a + 1 = 3b + 3 \Rightarrow 3a - 3b = 2$

$\therefore a - b = \frac{2}{3}$.

---

6. If f(x) = 3ax + b if x > 1; f(x) = 11 if x = 1; f(x) = 5ax − 2b if x < 1, is continuous at x = 1, find the values of a and b.

Solution :

RHL = 3a + b, LHL = 5a − 2b, f(1) = 11

For continuity, all three are equal:

$3a + b = 11$ …(1)

$5a - 2b = 11$ …(2)

(1) × 2 + (2):

$11a = 33 \Rightarrow a = 3, b = 11 - 9 = 2$

$\therefore a = 3, b = 2$.

---

7. Show that the function f(x) = |x − 3|, x ∈ R, is continuous but not differentiable at x = 3.

Solution : Continuity at 3:

$\text{LHL} = \lim_{x \to 3^-} (3 - x) = 0$, $\text{RHL} = \lim_{x \to 3^+} (x - 3) = 0$, $f(3) = 0$

So f is continuous at x = 3.

Differentiability at 3:

$$\text{LHD} = \lim_{h \to 0^-} \frac{|h| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\text{RHD} = \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

LHD ≠ RHD.

∴ f is continuous but not differentiable at x = 3.

---

8. Show that the function f(x) = |x − 1| + |x + 1|, x ∈ R, is not differentiable at the points x = −1 and x = 1.

Solution : Write f piecewise:

$f(x) = -2x$ for $x < -1$; $f(x) = 2$ for $-1 \le x \le 1$; $f(x) = 2x$ for $x > 1$

At x = 1:

LHD = derivative of 2 = 0, RHD = derivative of 2x = 2 ⇒ LHD ≠ RHD

At x = −1:

LHD = derivative of −2x = −2, RHD = derivative of 2 = 0 ⇒ LHD ≠ RHD

∴ f is not differentiable at x = −1 and x = 1.

---

9. If $y = x^{\cos x} + (\cos x)^{\sin x}$, find $\frac{dy}{dx}$.

Solution : Let $u = x^{\cos x}$ and $v = (\cos x)^{\sin x}$, so $y = u + v$.

Differentiate u by taking logs:

$$\log u = \cos x \log x \Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{\cos x}{x} - \sin x \log x$$

$$\frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right)$$

Differentiate v by taking logs:

$$\log v = \sin x \log \cos x \Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x \log \cos x - \sin x \tan x$$

$$\frac{dv}{dx} = (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x)$$

$$\therefore \frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x).$$

---

10. Differentiate $x^{\cos x} + \frac{x^2+1}{x^2-1}$ with respect to x.

Solution : First term (logarithmic differentiation):

$$u = x^{\cos x} \Rightarrow \log u = \cos x \log x \Rightarrow \frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right)$$

Second term (quotient rule):

$$\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$\therefore \text{Derivative} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) - \frac{4x}{(x^2-1)^2}.$$

---

11. If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$.

Solution : Take logarithms:

$$y \log x = x - y \Rightarrow y(1 + \log x) = x \Rightarrow y = \frac{x}{1 + \log x}$$

Differentiate (quotient rule):

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

Since $1 + \log x = \log e + \log x = \log(xe)$:

$$\therefore \frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}.$$

---

12. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Solution : Put x = sin A and y = sin B:

$\cos A+\cos B=a(\sin A-\sin B)$

$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2}=2a\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cot\frac{A-B}{2}=a\Rightarrow A-B=2\cot^{-1}a$$

So

$\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$ (a constant)

Differentiate with respect to x:

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\therefore\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$$

---

13. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ for $-1<x<1$, $x\neq y$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$.

Solution :

$$x\sqrt{1+y}=-y\sqrt{1+x}$$

Square both sides:

$x^2(1+y)=y^2(1+x)\Rightarrow x^2-y^2=xy(y-x)\Rightarrow(x-y)(x+y)=-xy(x-y)$

Since $x\neq y$, cancel $(x-y)$:

$$x+y=-xy\Rightarrow y(1+x)=-x\Rightarrow y=-\frac{x}{1+x}$$

Differentiate:

$$\frac{dy}{dx}=-\frac{(1+x)-x}{(1+x)^2}$$

$$\therefore\frac{dy}{dx}=-\frac{1}{(1+x)^2}.$$

---

14. If $y=e^{x+e^{x+e^{x+\cdots\infty}}}$, prove that $\frac{dy}{dx}=\frac{y}{1-y}$.

Solution : The exponent itself equals x + y:

$y=e^{x+y}\Rightarrow\log y=x+y$

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y}-1\right)=1 \Rightarrow \frac{dy}{dx}\cdot\frac{1-y}{y}=1$$

$$\therefore \frac{dy}{dx}=\frac{y}{1-y}.$$

---

15. Differentiate $\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$ with respect to x.

Solution : Write the expression as $\tan^{-1}\left(\frac{\sqrt{x}-x}{1+\sqrt{x}\cdot x}\right)$, which is of the form $\frac{A-B}{1+AB}$ with A = $\sqrt{x}$, B = x.

$$y=\tan^{-1}\sqrt{x}-\tan^{-1}x \left(\text{since } \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right)$$

Differentiate:

$$\frac{dy}{dx}=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{x}(1+x)}-\frac{1}{1+x^2}.$$

---

16. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, if $x \in (-1, 1)$.

Solution : Put $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

$$u=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

$$v=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x \text{ (as } 2\theta \in (-\pi/2, \pi/2))$$

Differentiate both with respect to x:

$$\frac{du}{dx}=\frac{1}{2(1+x^2)}, \frac{dv}{dx}=\frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv}=\frac{1}{4}.$$

---

17. Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$, for $\frac{1}{\sqrt{2}} < x < 1$.

Solution : Put $x = \cos\theta$; for $\frac{1}{\sqrt{2}} < x < 1$, $\theta \in \left(0, \frac{\pi}{4}\right)$.

$$u = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta \ (2\theta \in (0, \pi/2))$$

$$v = \sin^{-1}(2\cos\theta\sin\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

So u = v for these x.

$$\therefore \frac{du}{dv} = 1.$$

---

18. If $y = \sqrt{ax + b}$, prove that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.

Solution : Square: $y^2 = ax + b$. Differentiate:

$$2y\frac{dy}{dx} = a \Rightarrow y\frac{dy}{dx} = \frac{a}{2}$$

Differentiate again (product rule):

$$\frac{dy}{dx}\cdot\frac{dy}{dx} + y\frac{d^2y}{dx^2} = 0$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$

---

19. If $x = \sqrt{a^{\tan^{-1}t}}$ and $y = \sqrt{a^{\cot^{-1}t}}$, show that $x\frac{dy}{dx} + y = 0$.

Solution : Multiply x and y:

$$xy = \sqrt{a^{\tan^{-1}t + \cot^{-1}t}} = \sqrt{a^{\pi/2}}, \text{ a constant}$$

Differentiate xy = constant with respect to x:

$$x\frac{dy}{dx} + y = 0$$

$$\therefore x\frac{dy}{dx} + y = 0.$$

---

20. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Solution :

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta = -y, \ \frac{dy}{d\theta} = a\cos\theta + b\sin\theta = x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Differentiate again with respect to x:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} \Rightarrow y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx}$$

$$\therefore y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$

---

21. If $x = a\sin t - b\cos t$ and $y = a\cos t + b\sin t$, prove that $\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}$.

Solution :

$$\frac{dx}{dt} = a\cos t + b\sin t = y, \frac{dy}{dt} = -a\sin t + b\cos t = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Differentiate again with respect to x:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}.$$

---

22. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Solution :

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t$$

$$\frac{dy}{dx} = \tan t$$

Differentiate with respect to x using the chain rule:

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \frac{\sec^2 t}{at\cos t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}.$$

---

23. If $(a + bx)e^{y/x} = x$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

Solution : Take logs: $\frac{y}{x} = \log x - \log(a + bx)$, so

$$y = x\log x - x\log(a + bx)$$

$$\frac{dy}{dx} = \log x - \log(a+bx) + 1 - \frac{bx}{a+bx} = \frac{y}{x} + \frac{a}{a+bx}$$

$$x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad \ldots(1)$$

Differentiate (1) with respect to x:

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{a(a+bx) - abx}{(a+bx)^2} = \frac{a^2}{(a+bx)^2}$$

Multiply by x^2 and use (1):

$$\therefore x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 = \left(x\frac{dy}{dx} - y\right)^2.$$

---

24. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$.

Solution :

$$\frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2} \Rightarrow (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$

Differentiate again:

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$

Multiply by $(1+x^2)$:

$$\therefore (x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

---

25. If $y = \left(x + \sqrt{1+x^2}\right)^m$, prove that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

Solution : Differentiate using the chain rule:

$$\frac{dy}{dx} = m\left(x + \sqrt{1+x^2}\right)^{m-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= m\left(x + \sqrt{1+x^2}\right)^{m-1} \cdot \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

So $\sqrt{1+x^2}\, y_1 = my$, where $y_1 = \frac{dy}{dx}$. Squaring:

$$(1+x^2)y_1^2 = m^2y^2$$

Differentiate again w.r.t. x:

$$(1+x^2)\cdot 2y_1y_2 + 2xy_1^2 = 2m^2yy_1$$

Divide by $2y_1$ ($\neq 0$):

$$(1+x^2)y_2 + xy_1 = m^2y$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$$

---

26. Find the values of a and b so that f(x) = ax + b for x > −1 and f(x) = bx² − 3 for x ≤ −1 is differentiable for all values of x.

Solution : Both pieces are polynomials, so only x = −1 needs checking.

Continuity at x = −1:

$$b(-1)^2 - 3 = a(-1) + b \Rightarrow b - 3 = -a + b \Rightarrow a = 3$$

Differentiability at x = −1: LHD = 2bx at x = −1 = −2b; RHD = a.

$$-2b = a = 3 \Rightarrow b = -\frac{3}{2}$$

$$\therefore a = 3,\ b = -\frac{3}{2}.$$

---

27. Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$, where a and b are constants.

Solution : Let $u = y^x$, $v = x^y$, $w = x^x$; then $u' + v' + w' = 0$.

$$\log u = x\log y \Rightarrow u' = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right)$$

$$\log v = y\log x \Rightarrow v' = x^y\left(\frac{dy}{dx}\log x + \frac{y}{x}\right)$$

$$\log w = x\log x \Rightarrow w' = x^x(1+\log x)$$

Collecting the $\frac{dy}{dx}$ terms:

$$\therefore \frac{dy}{dx} = -\frac{y^x\log y + yx^{y-1} + x^x(1+\log x)}{xy^{x-1} + x^y\log x}.$$

---

28. If x cos(p + y) + cos p sin(p + y) = 0, prove that cos p $\frac{dy}{dx}$ = −cos²(p + y), where p is a constant.

Solution :

$$x = -\frac{\cos p\sin(p+y)}{\cos(p+y)} = -\cos p\tan(p+y)$$

Differentiate w.r.t. y:

$$\frac{dx}{dy} = -\cos p\sec^2(p+y)$$

$$\frac{dy}{dx} = -\frac{\cos^2(p+y)}{\cos p}$$

$\therefore \cos p \frac{dy}{dx} = -\cos^2(p+y).$

---

29. If $x^{30}y^{20} = (x+y)^{50}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Solution : Taking logarithms: $30 \log x + 20 \log y = 50 \log(x+y)$. Differentiate:

$$\frac{30}{x} + \frac{20}{y}\frac{dy}{dx} = \frac{50}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left(\frac{20}{y} - \frac{50}{x+y}\right) = \frac{50}{x+y} - \frac{30}{x}$$

$$\frac{dy}{dx} \cdot \frac{20x - 30y}{y(x+y)} = \frac{20x - 30y}{x(x+y)}$$

$\therefore \frac{dy}{dx} = \frac{y}{x}.$

---

30. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Solution :

$$\frac{dx}{dt} = ae^t(\sin t + \cos t) + ae^t(\cos t - \sin t) = 2ae^t \cos t$$

$$\frac{dy}{dt} = ae^t(\sin t - \cos t) + ae^t(\cos t + \sin t) = 2ae^t \sin t$$

$$\frac{dy}{dx} = \tan t$$

Also $x + y = 2ae^t \sin t$ and $x - y = 2ae^t \cos t$, so $\frac{x+y}{x-y} = \tan t$.

$\therefore \frac{dy}{dx} = \frac{x+y}{x-y}.$

---

31. If $\log\left(\sqrt{1+x^2} - x\right) = y\sqrt{1+x^2}$, show that $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$.

Solution : Differentiate the LHS w.r.t. x:

$$\frac{1}{\sqrt{1+x^2} - x}\left(\frac{x}{\sqrt{1+x^2}} - 1\right) = \frac{1}{\sqrt{1+x^2} - x} \cdot \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}} = -\frac{1}{\sqrt{1+x^2}}$$

Differentiate the RHS using the product rule:

$$\sqrt{1+x^2}\frac{dy}{dx} + \frac{xy}{\sqrt{1+x^2}}$$

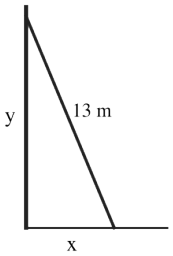
Equate the two and multiply by $\sqrt{1+x^2}$:

$$-1 = (1+x^2)\frac{dy}{dx} + xy$$

$$\therefore (1+x^2)\frac{dy}{dx} + xy + 1 = 0.$$

Chapter 6 – Application of Derivatives

1. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/s. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall?



Solution : Let the foot be x m from the wall and the top be y m high:

$$x^2 + y^2 = 169$$

Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When x = 5 m, y = 12 m, and $\frac{dx}{dt} = 2$ cm/s:

$$\frac{dy}{dt} = -\frac{5}{12} \times 2 = -\frac{5}{6} \text{ cm/s}$$

$\therefore$ The height is decreasing at $\frac{5}{6}$ cm/s.

---

2. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases when the side is 10 cm.

Solution : Area of an equilateral triangle of side x:

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}$$

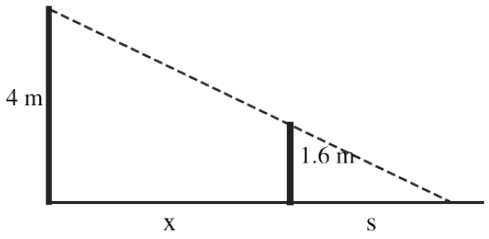
At x = 10 cm with $\frac{dx}{dt} = 2$ cm/s:

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3}$$

$\therefore$ The area increases at $10\sqrt{3}$ cm^2/s.

---

3. A man 1.6 m tall walks at the rate of 0.3 m/s away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening?



Solution : Let the man be x m from the lamp-post and his shadow be s m long. By similar triangles:

$$\frac{4}{x+s}=\frac{1.6}{s}\Rightarrow 4s=1.6x+1.6s\Rightarrow s=\frac{2}{3}x$$

Rate of lengthening of the shadow:

$$\frac{ds}{dt}=\frac{2}{3}\times 0.3=0.2\text{ m/s}$$

The tip is at distance x + s from the post:

$$\frac{d(x+s)}{dt}=0.3+0.2=0.5\text{ m/s}$$

$\therefore$ Tip moves at 0.5 m/s; shadow lengthens at 0.2 m/s.

---

4. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing.

Solution : For side x, the median (= altitude) is $m=\frac{\sqrt{3}}{2}x$.

$$\frac{dm}{dt}=\frac{\sqrt{3}}{2}\frac{dx}{dt}$$

$$2\sqrt{3}=\frac{\sqrt{3}}{2}\frac{dx}{dt}\Rightarrow\frac{dx}{dt}=4$$

$\therefore$ The side increases at 4 cm/s.

---

5. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base?

Solution : Let each equal side be a. Height $=\sqrt{a^2-\frac{b^2}{4}}$, so

$$A=\frac{b}{4}\sqrt{4a^2-b^2}$$

$$\frac{dA}{dt}=\frac{b}{4}\cdot\frac{8a}{2\sqrt{4a^2-b^2}}\frac{da}{dt}=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

At a = b with $\frac{da}{dt}=-3$ cm/s:

$$\frac{dA}{dt} = \frac{b^2}{\sqrt{3}\,b} \times (-3) = -\sqrt{3}\,b$$

$\therefore$ The area is decreasing at $\sqrt{3}\,b$ cm^2/s.

---

6. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the area of the first square.

Solution : Areas: $A_1 = x^2$ and $A_2 = y^2$.

$$\frac{dA_2}{dx} = 2y\frac{dy}{dx} = 2(x - x^2)(1 - 2x)$$

$$\frac{dA_1}{dx} = 2x$$

$$\frac{dA_2}{dA_1} = \frac{2x(1 - x)(1 - 2x)}{2x} = (1 - x)(1 - 2x)$$

$\therefore$ Required rate $= 2x^2 - 3x + 1$.

---

7. A particle moves along the curve $3y = ax^3 + 1$ such that at a point with x-coordinate 1, the y-coordinate is changing twice as fast as the x-coordinate. Find the value of a.

Solution : Differentiate with respect to t:

$$3\frac{dy}{dt} = 3ax^2\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = ax^2\frac{dx}{dt}$$

At $x = 1$, $\frac{dy}{dt} = 2\frac{dx}{dt}$:

$$2\frac{dx}{dt} = a\frac{dx}{dt}$$

$\therefore a = 2$.

---

8. Find the rate of change of the volume of a sphere with respect to its surface area when the radius is 12 cm.

Solution :

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dr} = 8\pi r$$

$$\frac{dV}{dS} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

At r = 12 cm:

$\therefore \frac{dV}{dS} = 6$ cm^3/cm^2.

---

9. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

Solution : Given $\frac{dV}{dt} = -kS$ ($k > 0$ constant), with $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$:

$$4\pi r^2 \frac{dr}{dt} = -k \cdot 4\pi r^2$$

$$\frac{dr}{dt} = -k$$

$\therefore$ The radius decreases at the constant rate k.

---

10. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (i) strictly increasing (ii) strictly decreasing.

Solution :

$f'(x) = x^3 - 3x^2 - 10x + 24 = (x - 2)(x + 3)(x - 4)$

Critical points: $x = -3, 2, 4$. Sign of $f'(x)$:

$x < -3$: $(-)(-)(-) < 0$; $-3 < x < 2$: $(-)(+)(-) > 0$

$2 < x < 4$: $(+)(+)(-) < 0$; $x > 4$: $(+)(+)(+) > 0$

$\therefore$ (i) Strictly increasing on $(-3, 2) \cup (4, \infty)$ (ii) Strictly decreasing on $(-\infty, -3) \cup (2, 4)$.

---

11. Find the intervals in which $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$, is strictly increasing or strictly decreasing.

Solution :

$$f'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Test the sign of f' in each interval:

$0 \le x < \frac{\pi}{4}$: $\cos x > \sin x \Rightarrow f' > 0$

$\frac{\pi}{4} < x < \frac{5\pi}{4}$: $f' < 0$

$\frac{5\pi}{4} < x \le 2\pi$: $f' > 0$

$\therefore$ Strictly increasing on $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$; strictly decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

---

12. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right)$.

Solution :

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$$

The denominator is always positive. For $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so the numerator is positive.

$f'(x) > 0$ on $\left(0, \frac{\pi}{4}\right)$

$\therefore$ f is strictly increasing in $\left(0, \frac{\pi}{4}\right)$.

---

13. Find the maximum and minimum values, if any, of $f(x) = -|x - 1| + 5$ for all $x \in R$.

Solution : $|x - 1| \geq 0$ for all x, so

$f(x) = 5 - |x - 1| \leq 5$, with equality at $x = 1$

As $|x| \to \infty$, $f(x) \to -\infty$, so f has no minimum.

$\therefore$ Maximum value = 5 (at $x = 1$); there is no minimum value.

---

14. Find the maximum and minimum values of $f(x) = 5 + \sin 2x$.

Solution : For all x, $-1 \leq \sin 2x \leq 1$:

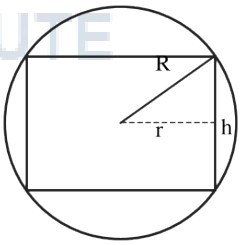
$4 \leq 5 + \sin 2x \leq 6$

Maximum 6 when $\sin 2x = 1$ ($x = \frac{\pi}{4}$); minimum 4 when $\sin 2x = -1$ ($x = \frac{3\pi}{4}$)

$\therefore$ Maximum value = 6, minimum value = 4.

---

15. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.



Solution : Let the cylinder have radius r and height h. From the right triangle through the centre:

$$r^2 + \frac{h^2}{4} = R^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}$$

$$V = \pi r^2 h = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

$$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$$

$$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2} < 0 \Rightarrow \text{maximum}$$

Maximum volume:

$$r^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3},\ V = \pi \cdot \frac{2R^2}{3} \cdot \frac{2R}{\sqrt{3}}$$

$\therefore h = \frac{2R}{\sqrt{3}}$ and maximum volume $= \frac{4\pi R^3}{3\sqrt{3}}$ cubic units.

---

16. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$.

Solution : Let the slant height be l and the semi-vertical angle θ:

$r = l\sin\theta,\ h = l\cos\theta$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

$$\frac{dV}{d\theta} = \frac{\pi l^3}{3}(2\sin\theta\cos^2\theta - \sin^3\theta) = \frac{\pi l^3}{3}\sin\theta(2\cos^2\theta - \sin^2\theta)$$

$$\frac{dV}{d\theta} = 0 \Rightarrow \tan^2\theta = 2 \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

dV/dθ changes from positive to negative at this θ, so V is maximum.

$\therefore$ Semi-vertical angle $= \cos^{-1}\frac{1}{\sqrt{3}}$.

---

17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

Solution : Let the side of the base be x and the depth h; the volume $V = x^2h$ is fixed.

Surface area (open top):

$$S = x^2 + 4xh = x^2 + \frac{4V}{x}$$

$$\frac{dS}{dx} = 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V = 2x^2h \Rightarrow x = 2h$$

$$\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0 \Rightarrow \text{minimum}$$

Cost is proportional to the area, so it is least when x = 2h.

$\therefore$ Depth $= \frac{1}{2} \times$ width.

---

18. A tank with rectangular base and rectangular sides, open at the top, is to be constructed so that its depth is 3 m and volume is 75 m^3. If building the tank costs ₹ 100 per m^2 for the base and ₹ 50 per m^2 for the sides, find the cost of the least expensive tank.

Solution : Let the base be $l \times b$. Then $3lb = 75$, so $lb = 25$.

$$\text{Cost } C = 100(lb) + 50 \times 2 \times 3(l + b) = 2500 + 300(l + b)$$

With $b = \frac{25}{l}$:

$$C = 2500 + 300\left(l + \frac{25}{l}\right)$$

$$\frac{dC}{dl} = 300\left(1 - \frac{25}{l^2}\right) = 0 \Rightarrow l = 5,\ b = 5$$

$$\frac{d^2C}{dl^2} = \frac{15000}{l^3} > 0 \Rightarrow \text{minimum}$$

$$C = 2500 + 300 \times 10 = 5500$$

$\therefore$ Least cost = ₹ 5500.

---

19. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Solution : Let the side be x, hypotenuse h, with $x + h = k$ (constant). The third side is $\sqrt{h^2 - x^2} = \sqrt{k^2 - 2kx}$.

$$A = \frac{1}{2}x\sqrt{k^2 - 2kx} \Rightarrow A^2 = \frac{1}{4}(k^2x^2 - 2kx^3)$$

$$\frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) = 0 \Rightarrow x = \frac{k}{3}$$

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12kx) = -\frac{k^2}{2} < 0 \text{ at } x = k/3 \Rightarrow \text{maximum}$$

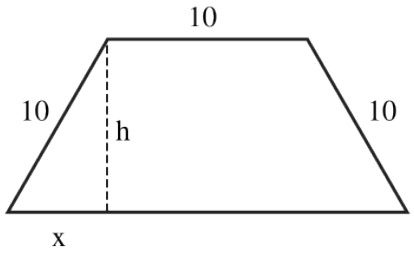
Then $h = \frac{2k}{3}$, and the angle θ between them satisfies

$$\cos\theta = \frac{x}{h} = \frac{1}{2}$$

$\therefore$ Area is maximum when $\theta = \frac{\pi}{3}$.

---

20. If the lengths of three sides of a trapezium other than the base are equal to 10 cm, find the area of the trapezium when it is maximum.



Solution : Drop perpendiculars of height h from the ends of the top side; let each slanting side project x on the base. Then

$$\text{base} = 10 + 2x,\ h = \sqrt{100 - x^2}$$

$$A = \frac{1}{2}(10 + 10 + 2x)\sqrt{100 - x^2} = (10 + x)\sqrt{100 - x^2}$$

$$\frac{dA}{dx} = \sqrt{100 - x^2} - \frac{x(10 + x)}{\sqrt{100 - x^2}} = \frac{100 - 10x - 2x^2}{\sqrt{100 - x^2}}$$

$$\frac{dA}{dx} = 0 \Rightarrow x^2 + 5x - 50 = 0 \Rightarrow x = 5$$

dA/dx changes from + to − at x = 5, so A is maximum:

$$A = 15 \times \sqrt{75} = 15 \times 5\sqrt{3}$$

∴ Maximum area = $75\sqrt{3}$ cm².

---

21. Of all the closed right circular cylindrical cans of volume 128π cm³, find the dimensions of the can which has minimum surface area.

Solution :

$$\pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2}$$

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{256\pi}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4$$

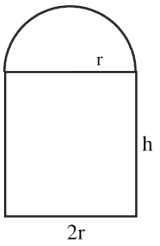
$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3} > 0 \Rightarrow \text{minimum}$$

$$h = \frac{128}{16} = 8$$

∴ Radius = 4 cm, height = 8 cm.

---

22. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.



Solution : Let the rectangle be 2r wide and h high (semicircle of radius r on top).

Perimeter: $2r + 2h + \pi r = 10 \Rightarrow h = \dfrac{10 - (\pi + 2)r}{2}$

Area $A = 2rh + \dfrac{1}{2}\pi r^2 = 10r - (\pi + 2)r^2 + \dfrac{\pi}{2}r^2 = 10r - \left(2 + \dfrac{\pi}{2}\right)r^2$

$\dfrac{dA}{dr} = 10 - (4 + \pi)r = 0 \Rightarrow r = \dfrac{10}{\pi + 4}$

$\dfrac{d^2A}{dr^2} = -(4 + \pi) < 0 \Rightarrow$ maximum

$h = \dfrac{10(\pi + 4) - 10(\pi + 2)}{2(\pi + 4)} = \dfrac{10}{\pi + 4}$

$\therefore$ Width of the rectangle $= \dfrac{20}{\pi + 4}$ m, height $= \dfrac{10}{\pi + 4}$ m (radius of semicircle $\dfrac{10}{\pi + 4}$ m).

---

23. A wire of length 28 m is to be cut into two pieces. One piece is made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area is minimum?

Solution : Let the square piece be x m; the circle piece is 28 – x.

Side $= \dfrac{x}{4}$, radius $r = \dfrac{28 - x}{2\pi}$

$A = \dfrac{x^2}{16} + \pi\dfrac{(28 - x)^2}{4\pi^2} = \dfrac{x^2}{16} + \dfrac{(28 - x)^2}{4\pi}$

$\dfrac{dA}{dx} = \dfrac{x}{8} - \dfrac{28 - x}{2\pi} = 0 \Rightarrow \pi x = 4(28 - x) \Rightarrow x = \dfrac{112}{\pi + 4}$

$\dfrac{d^2A}{dx^2} = \dfrac{1}{8} + \dfrac{1}{2\pi} > 0 \Rightarrow$ minimum

$\therefore$ Square piece $= \dfrac{112}{\pi + 4}$ m, circle piece $= \dfrac{28\pi}{\pi + 4}$ m.

---

24. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point (1, 4).

Solution : Take a point $\left(\dfrac{y^2}{2}, y\right)$ on the curve. Square of the distance:

$D = \left(\dfrac{y^2}{2} - 1\right)^2 + (y - 4)^2$

$$\frac{dD}{dy} = 2\left(\frac{y^2}{2} - 1\right)y + 2(y-4) = y^3 - 8 = 0 \Rightarrow y = 2$$

$$\frac{d^2D}{dy^2} = 3y^2 = 12 > 0 \Rightarrow \text{minimum}$$

$$x = \frac{4}{2} = 2$$

∴ The required point is (2, 2).

---

25. If $f(x) = \frac{1}{4x^2 + 2x + 1}$, $x \in R$, find the maximum value of f(x).

Solution : f is maximum when the denominator $g(x) = 4x^2 + 2x + 1$ is minimum ($g(x) > 0$ for all x).

Method 1 (completing the square):

$$g(x) = 4\left(x^2 + \frac{x}{2} + \frac{1}{16}\right) + 1 - \frac{1}{4} = 4\left(x + \frac{1}{4}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$$

Method 2 (derivative):

$g'(x) = 8x + 2 = 0 \Rightarrow x = -\frac{1}{4}$; $g''(x) = 8 > 0$, so g is minimum at $x = -\frac{1}{4}$

$$g\left(-\frac{1}{4}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$$

∴ Maximum value of $f(x) = \frac{1}{3/4} = \frac{4}{3}$ (at $x = -\frac{1}{4}$).

---

26. If f(x) = a(tan x − cot x), where a > 0, find whether f(x) is increasing or decreasing in its domain.

Solution : f is defined wherever tan x and cot x are defined, i.e. $x \neq \frac{n\pi}{2}$, $n \in Z$.

Differentiate:

$f'(x) = a(\sec^2 x + \operatorname{cosec}^2 x)$

$\sec^2 x \geq 1$ and $\operatorname{cosec}^2 x \geq 1$ for every x in the domain, and $a > 0$:

$f'(x) \geq 2a > 0$

∴ f(x) is (strictly) increasing in its domain.

Note: Case-study questions are recommended from this chapter for **CBSE Board 2026-27**.
Practise them here: **Class XII Case Studies (click here)**

Chapter 7 – Integrals

1. Find: $\int \frac{\sin x}{\sin(x-a)}\,dx$.

Solution : Put x − a = t, so x = t + a and dx = dt:

$$\int \frac{\sin(t+a)}{\sin t}dt = \int \frac{\sin t\cos a + \cos t\sin a}{\sin t}dt$$
$$= \int(\cos a + \sin a\cot t)\,dt = t\cos a + \sin a\log|\sin t| + C$$

Substitute back t = x − a:

$$\therefore \int \frac{\sin x}{\sin(x-a)}dx = (x-a)\cos a + \sin a\log|\sin(x-a)| + C.$$

---

2. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$.

Solution : Put x + a = t, so x − a = t − 2a and dx = dt:

$$\int \frac{\sin(t-2a)}{\sin t}dt = \int \frac{\sin t\cos 2a - \cos t\sin 2a}{\sin t}dt$$
$$= \int(\cos 2a - \sin 2a\cot t)\,dt = t\cos 2a - \sin 2a\log|\sin t| + C_1$$

Substitute back (the constant a cos 2a merges with C):

$$\therefore \int \frac{\sin(x-a)}{\sin(x+a)}dx = x\cos 2a - \sin 2a\log|\sin(x+a)| + C.$$

---

3. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$.

Solution : Put sin x − cos x = t. Then (cos x + sin x) dx = dt and

$$t^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$$

Limits: $x = \pi/6 \Rightarrow t = \frac{1-\sqrt{3}}{2}$; $x = \pi/3 \Rightarrow t = \frac{\sqrt{3}-1}{2}$

$$I = \int_{-k}^{k} \frac{dt}{\sqrt{1-t^2}}, \text{ where } k = \frac{\sqrt{3}-1}{2}$$

$$I = [\sin^{-1} t]_{-k}^{k} = 2\sin^{-1} k$$

$$\therefore I = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

---

4. Evaluate: $\int \frac{dx}{\sqrt{5-4x-2x^2}}$.

Solution : Complete the square:

$$5-4x-2x^2=-2\left(x^2+2x-\frac{5}{2}\right)=2\left[\frac{7}{2}-(x+1)^2\right]$$

$$I=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{7/2}\right)^2-(x+1)^2}}$$

Use $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$:

$$\therefore I=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)+C.$$

---

5. Evaluate: $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$.

Solution : Write the numerator as a multiple of the derivative of the quadratic plus a constant:

$$x+2=\frac{1}{2}(2x+2)+1$$

$$I=\frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}dx+\int\frac{dx}{\sqrt{(x+1)^2+2}}$$

First integral: put $x^2+2x+3=t\Rightarrow\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\sqrt{t}$

Second integral: $\int\frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|$

$$\therefore I=\sqrt{x^2+2x+3}+\log\left|x+1+\sqrt{x^2+2x+3}\right|+C.$$

---

6. Find: $\int\frac{dx}{x(x^5+3)}$.

Solution : Multiply numerator and denominator by x^4:

$$I=\int\frac{x^4dx}{x^5(x^5+3)}$$

Put $x^5=t$, $5x^4dx=dt$:

$$I=\frac{1}{5}\int\frac{dt}{t(t+3)}=\frac{1}{5}\cdot\frac{1}{3}\int\left(\frac{1}{t}-\frac{1}{t+3}\right)dt$$

$$=\frac{1}{15}\log\left|\frac{t}{t+3}\right|+C$$

$$\therefore I=\frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right|+C.$$

---

7. Find: $\int \frac{dx}{x(x+1)(x+2)}$.

Solution : Partial fractions:

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

$x = 0: 1 = 2A \Rightarrow A = \frac{1}{2}$; $x = -1: 1 = -B \Rightarrow B = -1$; $x = -2: 1 = 2C \Rightarrow C = \frac{1}{2}$

Integrate term by term:

$$\therefore I = \frac{1}{2}\log|x| - \log|x+1| + \frac{1}{2}\log|x+2| + C.$$

---

8. Evaluate: $\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$.

Solution : Put $x^2 = y$ (only for the partial fractions):

$$\frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9}$$

$y = -4: -4 = 5A \Rightarrow A = -\frac{4}{5}$; $y = -9: -9 = -5B \Rightarrow B = \frac{9}{5}$

$$I = -\frac{4}{5}\int \frac{dx}{x^2+4} + \frac{9}{5}\int \frac{dx}{x^2+9}$$

$$\therefore I = -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C.$$

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9. Find: $\int \frac{x^2+x+1}{(x+2)(x^2+1)}\,dx$.

Solution :

$$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$x = -2: 3 = 5A \Rightarrow A = \frac{3}{5}$

Compare x^2: $1 = A + B \Rightarrow B = \frac{2}{5}$; constants: $1 = A + 2C \Rightarrow C = \frac{1}{5}$

$$I = \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{dx}{x^2+1}$$

$$\therefore I = \frac{3}{5}\log|x+2| + \frac{1}{5}\log(x^2+1) + \frac{1}{5}\tan^{-1}x + C.$$

---

10. Evaluate: $\int \frac{\tan\theta + \tan^3\theta}{1 + \tan^3\theta}\, d\theta.$

Solution : Numerator = $\tan\theta(1 + \tan^2\theta) = \tan\theta \sec^2\theta$. Put $\tan\theta = t$, $\sec^2\theta\, d\theta = dt$:

$$I = \int \frac{t}{1+t^3} dt = \int \frac{t}{(1+t)(t^2 - t + 1)} dt$$

Partial fractions: $\frac{t}{(1+t)(t^2-t+1)} = -\frac{1}{3(1+t)} + \frac{t+1}{3(t^2-t+1)}$

$$\int \frac{t+1}{t^2-t+1} dt = \frac{1}{2}\log(t^2 - t + 1) + \frac{3}{2}\int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{2}\log(t^2 - t + 1) + \sqrt{3}\tan^{-1}\frac{2t-1}{\sqrt{3}}$$

Combine, with $t = \tan\theta$:

$$\therefore I = -\frac{1}{3}\log|1 + \tan\theta| + \frac{1}{6}\log(\tan^2\theta - \tan\theta + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C.$$

---

11. Find: $\int \frac{\log x}{(x+1)^2}\, dx.$

Solution : Integrate by parts, taking $\log x$ as the first function:

$$I = \log x \cdot \left(-\frac{1}{x+1}\right) - \int \frac{1}{x} \cdot \left(-\frac{1}{x+1}\right) dx$$

$$= -\frac{\log x}{x+1} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$\therefore I = -\frac{\log x}{x+1} + \log\left|\frac{x}{x+1}\right| + C.$$

---

12. Evaluate: $\int \frac{(x-4)e^x}{(x-2)^3}\, dx.$

Solution : Split the fraction to get the form $e^x[f(x) + f'(x)]$:

$$\frac{x-4}{(x-2)^3} = \frac{(x-2)-2}{(x-2)^3} = \frac{1}{(x-2)^2} - \frac{2}{(x-2)^3}$$

With $f(x) = \frac{1}{(x-2)^2}$, $f'(x) = -\frac{2}{(x-2)^3}$. Use $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$:

$$\therefore I = \frac{e^x}{(x-2)^2} + C.$$

---

13. Evaluate: $\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx.$

Solution : Use $1 - \cos 4x = 2\sin^2 2x$ and $\sin 4x = 2\sin 2x \cos 2x$:

$$\frac{\sin 4x - 4}{1 - \cos 4x} = \frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x} = \cot 2x - 2\,\text{cosec}^2 2x$$

With $f(x) = \cot 2x$, $f'(x) = -2\text{cosec}^2 2x$:

$\therefore I = e^x \cot 2x + C.$

---

14. Find: $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx$.

Solution : Use $1 + \cos x = 2\cos^2 \frac{x}{2}$ and $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$:

$$\frac{1 + \sin x}{1 + \cos x} = \frac{1}{2}\sec^2 \frac{x}{2} + \tan \frac{x}{2}$$

With $f(x) = \tan \frac{x}{2}$, $f'(x) = \frac{1}{2}\sec^2 \frac{x}{2}$:

$\therefore I = e^x \tan \frac{x}{2} + C.$

---

15. Given $\int e^x (\tan x + 1)\sec x\, dx = e^x f(x) + C$, write f(x).

Solution :

$(\tan x + 1)\sec x = \sec x + \sec x \tan x$

With $f(x) = \sec x$, $f'(x) = \sec x \tan x$, so the integrand is $e^x[f(x) + f'(x)]$:

$\int e^x(\sec x + \sec x \tan x)dx = e^x \sec x + C$

$\therefore f(x) = \sec x.$

---

16. Evaluate: $\int e^{2x} \sin(3x + 1)\, dx$.

Solution : Let $I = \int e^{2x}\sin(3x + 1)dx$. Integrate by parts twice (trigonometric function first):

$$I = -\frac{1}{3}e^{2x}\cos(3x + 1) + \frac{2}{3}\int e^{2x}\cos(3x + 1)dx$$

$$\int e^{2x}\cos(3x + 1)dx = \frac{1}{3}e^{2x}\sin(3x + 1) - \frac{2}{3}I$$

Substitute:

$$I = -\frac{1}{3}e^{2x}\cos(3x + 1) + \frac{2}{9}e^{2x}\sin(3x + 1) - \frac{4}{9}I$$

$$\frac{13}{9}I = \frac{e^{2x}}{9}[2\sin(3x + 1) - 3\cos(3x + 1)]$$

$$\therefore I = \frac{e^{2x}}{13}[2\sin(3x + 1) - 3\cos(3x + 1)] + C.$$

---

17. Find: $\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x\, dx$.

Solution : Put $5^{5^{5^x}} = t$. Differentiate step by step:

$$dt = 5^{5^{5^x}} \log 5 \cdot 5^{5^x} \log 5 \cdot 5^x \log 5\, dx$$

$$5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x dx = \frac{dt}{(\log 5)^3}$$

$$I = \frac{1}{(\log 5)^3}\int dt = \frac{t}{(\log 5)^3} + C$$

$$\therefore I = \frac{5^{5^{5^x}}}{(\log 5)^3} + C.$$

---

18. Find: $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$.

Solution : Put $x = \cos^2\theta$, $dx = -2\sin\theta\cos\theta\, d\theta$ $(0 < \theta < \pi/2)$:

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan\frac{\theta}{2}$$

$$I = -\int \tan\frac{\theta}{2} \cdot 2\sin\theta\cos\theta\, d\theta = -\int 4\sin^2\frac{\theta}{2}\cos\theta\, d\theta = -\int 2(1-\cos\theta)\cos\theta\, d\theta$$

$$= -2\sin\theta + \int(1+\cos 2\theta)d\theta = -2\sin\theta + \theta + \frac{1}{2}\sin 2\theta + C$$

Back-substitute: $\cos\theta = \sqrt{x}$, $\sin\theta = \sqrt{1-x}$, $\theta = \cos^{-1}\sqrt{x}$, $\frac{1}{2}\sin 2\theta = \sqrt{x-x^2}$:

$$\therefore I = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C.$$

---

19. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$.

Solution : Write $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ …(1)

Use $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ with $a + b = \pi/2$:

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \quad \ldots(2)$$

Add (1) and (2):

$$2I = \int_{\pi/6}^{\pi/3} 1\, dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore I = \frac{\pi}{12}.$$

---

20. Evaluate: $\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}\,dx$.

Solution : $I = \int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx$ …(1). Replace x by $1 + 3 - x = 4 - x$:

$$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}}dx \text{ …(2)}$$

Add (1) and (2):

$$2I = \int_1^3 dx = 2$$

$\therefore I = 1.$

---

21. Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

Solution : $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$. Replace x by $\pi - x$:

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

Add the two forms:

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t$, $-\sin x\,dx = dt$; $x = 0 \Rightarrow t = 1$, $x = \pi \Rightarrow t = -1$:

$$2I = \pi\int_{-1}^{1} \frac{dt}{1+t^2} = \pi[\tan^{-1} t]_{-1}^{1} = \pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi^2}{2}$$

$\therefore I = \frac{\pi^2}{4}$.

---

22. Evaluate: $\int_0^{\pi/4} \log(1+\tan x)\,dx$.

Solution : Replace x by $\frac{\pi}{4} - x$:

$$1 + \tan\left(\frac{\pi}{4} - x\right) = 1 + \frac{1-\tan x}{1+\tan x} = \frac{2}{1+\tan x}$$

$$I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]dx = \frac{\pi}{4}\log 2 - I$$

$$2I = \frac{\pi}{4}\log 2$$

$\therefore I = \frac{\pi}{8}\log 2.$

---

23. Evaluate: $\int_0^{\pi/2}[\log(\sin x) - \log(2\cos x)]\,dx$.

Solution :

$$I = \int_0^{\pi/2}\log\sin x\,dx - \int_0^{\pi/2}\log\cos x\,dx - \int_0^{\pi/2}\log 2\,dx$$

By the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$\int_0^{\pi/2}\log\cos x\,dx = \int_0^{\pi/2}\log\sin x\,dx$$

So the first two integrals cancel:

$$\therefore I = -\frac{\pi}{2}\log 2.$$

---

24. Evaluate: $\int_0^{\pi/2}|\sin x - \cos x|\,dx$.

Solution : $\sin x - \cos x < 0$ on $\left(0, \frac{\pi}{4}\right)$ and > 0 on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$:

$$I = \int_0^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2}(\sin x - \cos x)dx$$
$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$
$$= (\sqrt{2} - 1) + (-1 + \sqrt{2})$$

$$\therefore I = 2(\sqrt{2} - 1).$$

---

25. Evaluate: $\int_{-5}^{5}|x + 2|\,dx$.

Solution : $|x + 2| = -(x + 2)$ for $x < -2$ and $x + 2$ for $x \geq -2$:

$$I = \int_{-5}^{-2} -(x + 2)dx + \int_{-2}^{5}(x + 2)dx$$
$$= \left[-\frac{(x+2)^2}{2}\right]_{-5}^{-2} + \left[\frac{(x+2)^2}{2}\right]_{-2}^{5} = \frac{9}{2} + \frac{49}{2}$$

$$\therefore I = 29.$$

---

26. Evaluate: $\int_{-a}^{a} f(x)\,dx$, where $f(x) = \frac{9^x}{1 + 9^x}$.

Solution : Use $\int_{-a}^{a}f(x)dx = \int_{-a}^{a}f(-x)dx$:

$$f(-x) = \frac{9^{-x}}{1 + 9^{-x}} = \frac{1}{9^x + 1}$$

$$f(x) + f(-x) = \frac{9^x + 1}{1 + 9^x} = 1$$

$$2I = \int_{-a}^{a}[f(x) + f(-x)]dx = \int_{-a}^{a}1\,dx = 2a$$

$$\therefore I = a.$$

---

27. Evaluate: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx$.

Solution : Replace x by $\frac{\pi}{2} - x$: $I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}dx$. Add:

$$2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|\right]_0^{\pi/2} = \frac{1}{\sqrt{2}}\left[\log\tan\frac{3\pi}{8} - \log\tan\frac{\pi}{8}\right]$$

$$\tan\frac{3\pi}{8} = \sqrt{2}+1,\ \tan\frac{\pi}{8} = \sqrt{2}-1 \Rightarrow 2I = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\therefore I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

---

28. Find: $\int \frac{dx}{\sin x + \sin 2x}$.

Solution :

$$\int \frac{dx}{\sin x(1+2\cos x)} = \int \frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$$

Put $\cos x = t$, $-\sin x\,dx = dt$:

$$-\int \frac{dt}{(1-t)(1+t)(1+2t)};\ \frac{1}{(1-t)(1+t)(1+2t)} = \frac{1/6}{1-t} - \frac{1/2}{1+t} + \frac{4/3}{1+2t}$$

$$= \frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2t| + C$$

$$\therefore = \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + C.$$

---

29. Find: $\int \frac{dx}{\sqrt{\sin^3 x\cos(x-\alpha)}}$.

Solution :

$\sin^3 x\cos(x-\alpha) = \sin^3 x(\cos x\cos\alpha + \sin x\sin\alpha) = \sin^4 x(\cot x\cos\alpha + \sin\alpha)$

$$\text{Integral} = \int \frac{\text{cosec}^2 x\,dx}{\sqrt{\cos\alpha\cot x + \sin\alpha}}$$

Put $\cos\alpha\cot x + \sin\alpha = t$, $-\cos\alpha\,\text{cosec}^2 x\,dx = dt$:

$$-\frac{1}{\cos\alpha}\int t^{-1/2}dt = -\frac{2\sqrt{t}}{\cos\alpha}$$

$$\therefore = -\frac{2}{\cos\alpha}\sqrt{\cos\alpha\cot x + \sin\alpha} + C.$$

---

30. Evaluate: $\int_{1/3}^{1}\frac{(x-x^3)^{1/3}}{x^4}\,dx$.

Solution :

$$\frac{(x-x^3)^{1/3}}{x^4}=\frac{x\left(\frac{1}{x^2}-1\right)^{1/3}}{x^4}=\frac{1}{x^3}\left(\frac{1}{x^2}-1\right)^{1/3}$$

Put $\frac{1}{x^2}-1=t$, $-\frac{2}{x^3}dx=dt$; $x=\frac{1}{3}\Rightarrow t=8$, $x=1\Rightarrow t=0$:

$$I=-\frac{1}{2}\int_8^0 t^{1/3}dt=\frac{1}{2}\cdot\frac{3}{4}[t^{4/3}]_0^8=\frac{3}{8}\times 16$$

$\therefore I=6.$

---

31. Evaluate: $\int_{\log\sqrt{2}}^{\log\sqrt{3}}\frac{dx}{(e^x+e^{-x})(e^x-e^{-x})}$.

Solution : Simplify the integrand:

$(e^x+e^{-x})(e^x-e^{-x})=e^{2x}-e^{-2x}$, so the integrand $=\frac{1}{e^{2x}-e^{-2x}}=\frac{e^{2x}}{e^{4x}-1}$

Put $e^{2x}=t$, $2e^{2x}dx=dt$. Limits: $x=\log\sqrt{2}\Rightarrow t=2$; $x=\log\sqrt{3}\Rightarrow t=3$.

$$I=\frac{1}{2}\int_2^3\frac{dt}{t^2-1}=\frac{1}{2}\cdot\frac{1}{2}\left[\log\left|\frac{t-1}{t+1}\right|\right]_2^3$$

$$=\frac{1}{4}\left[\log\frac{2}{4}-\log\frac{1}{3}\right]=\frac{1}{4}\log\left(\frac{1}{2}\times 3\right)$$

$\therefore I=\frac{1}{4}\log\frac{3}{2}.$

---

32. Evaluate: $\int_1^e\frac{dx}{\sqrt{4x^2-(x\log x)^2}}$.

Solution : Take x out of the square root ($x>0$ on $[1, e]$):

$\sqrt{4x^2-x^2(\log x)^2}=x\sqrt{4-(\log x)^2}$, so $I=\int_1^e\frac{dx}{x\sqrt{4-(\log x)^2}}$

Put $\log x=t$, $\frac{dx}{x}=dt$. Limits: $x=1\Rightarrow t=0$; $x=e\Rightarrow t=1$.

$$I=\int_0^1\frac{dt}{\sqrt{4-t^2}}=\left[\sin^{-1}\frac{t}{2}\right]_0^1=\sin^{-1}\frac{1}{2}-0$$

$\therefore I=\frac{\pi}{6}.$

---

33. Evaluate: $\int_0^{2\pi} \frac{dx}{1+e^{\sin x}}$.

Solution : Let I be the integral. Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ with a = 2π, and sin(2π − x) = −sin x:

$$I = \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}} = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}dx$$

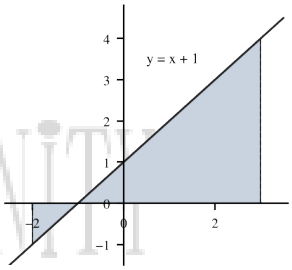
Adding the two forms of I:

$$2I = \int_0^{2\pi} \frac{1+e^{\sin x}}{1+e^{\sin x}}dx = \int_0^{2\pi} 1\,dx = 2\pi$$

$\therefore I = \pi$.

Chapter 8 – Application of Integrals

1. Using integration, find the area of the region bounded by the line x − y + 1 = 0, the lines x = −2, x = 3 and the x-axis.



Solution : The line is y = x + 1; it is below the x-axis for −2 ≤ x < −1 and above it for −1 < x ≤ 3.

$$\text{Area} = \left|\int_{-2}^{-1}(x+1)dx\right| + \int_{-1}^{3}(x+1)dx$$

$$\int_{-2}^{-1}(x+1)dx = \left[\frac{(x+1)^2}{2}\right]_{-2}^{-1} = 0 - \frac{1}{2} = -\frac{1}{2}$$

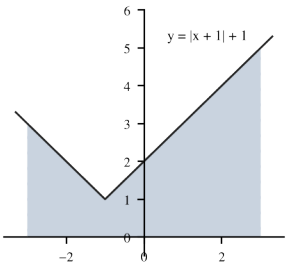
$$\int_{-1}^{3}(x+1)dx = \left[\frac{(x+1)^2}{2}\right]_{-1}^{3} = 8$$

$$\text{Area} = \frac{1}{2} + 8$$

$\therefore$ Required area = $\frac{17}{2}$ sq. units.

---

2. Using integration, find the area of the region bounded by the curve y = |x + 1| + 1, the lines x = −3, x = 3 and y = 0.



Solution : y = −x for x < −1 and y = x + 2 for x ≥ −1 (always positive).

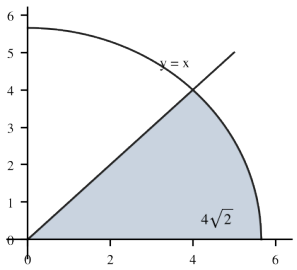
$$\text{Area} = \int_{-3}^{-1}(-x)dx + \int_{-1}^{3}(x+2)dx$$

$$= \left[-\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$
$$= \left(-\frac{1}{2} + \frac{9}{2}\right) + \left(\frac{21}{2} + \frac{3}{2}\right) = 4 + 12$$

∴ Required area = 16 sq. units.

---

3. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$.



Solution : Intersection of y = x with the circle: $2x^2 = 32 \Rightarrow x = 4$, so the point is (4, 4). The circle meets the x-axis at $x = 4\sqrt{2}$.

$$\text{Area} = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$\int_0^4 x\,dx = 8$$

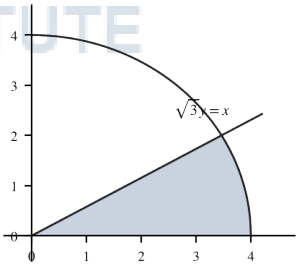
$$\int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx = \left[\frac{x}{2}\sqrt{32 - x^2} + 16\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}} = 8\pi - (8 + 4\pi) = 4\pi - 8$$

Area = $8 + 4\pi - 8$

∴ Required area = 4π sq. units.

---

4. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.



Solution : Intersection: $x^2 + \frac{x^2}{3} = 16 \Rightarrow x = 2\sqrt{3}$, y = 2. The region lies between the x-axis, the line and the circle.

$$\text{Area} = \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^4 \sqrt{16 - x^2}\,dx$$

$$\int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx = \frac{12}{2\sqrt{3}} = 2\sqrt{3}$$

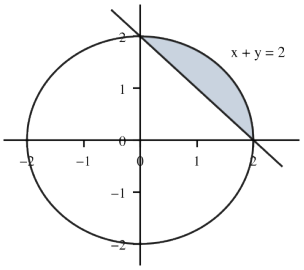
$$\int_{2\sqrt{3}}^4 \sqrt{16 - x^2}\,dx = \left[\frac{x}{2}\sqrt{16 - x^2} + 8\sin^{-1}\frac{x}{4}\right]_{2\sqrt{3}}^4 = 4\pi - \left(2\sqrt{3} + \frac{8\pi}{3}\right)$$

$$\text{Area} = 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$\therefore$ Required area $= \frac{4\pi}{3}$ sq. units.

---

5. Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$.



Solution : The line $x + y = 2$ cuts the circle at (2, 0) and (0, 2); the region is the minor segment.

$$\text{Area} = \int_0^2 \sqrt{4 - x^2}\,dx - \int_0^2 (2 - x)dx$$

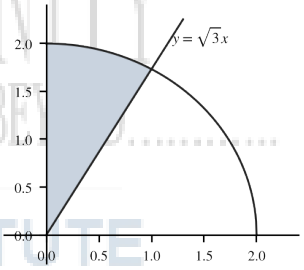
$$\int_0^2 \sqrt{4 - x^2}\,dx = \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = \pi$$

$$\int_0^2 (2 - x)dx = 2$$

$\therefore$ Required area $= (\pi - 2)$ sq. units.

---

6. Using integration, find the area of the region in the first quadrant bounded by the line $y = \sqrt{3}x$, the curve $y = \sqrt{4 - x^2}$ and the y-axis.



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Solution : Intersection: $3x^2 = 4 - x^2 \Rightarrow x = 1, y = \sqrt{3}$. Between $x = 0$ and $x = 1$ the circle lies above the line.

$$\text{Area} = \int_0^1 \left(\sqrt{4 - x^2} - \sqrt{3}x\right)dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^1 - \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$\therefore$ Required area $= \frac{\pi}{3}$ sq. units.

---

7. Using integration, find the area of the triangular region whose sides have equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

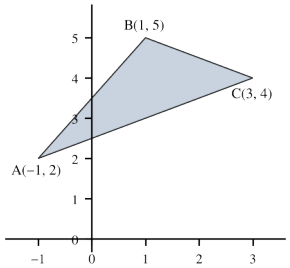
Solution : The lines $y = 2x + 1$ and $y = 3x + 1$ meet at (0, 1); $x = 4$ meets them at (4, 9) and (4, 13).

$$\text{Area} = \int_0^4 [(3x+1)-(2x+1)]dx = \int_0^4 x\,dx$$

$\therefore$ Required area $= \left[\frac{x^2}{2}\right]_0^4 = 8$ sq. units.

---

8. Using integration, find the area of the region bounded by the triangle whose vertices are (−1, 2), (1, 5) and (3, 4).



Solution : Equations of the sides:

$$\text{AB: } y = \frac{3x+7}{2};\ \text{BC: } y = \frac{11-x}{2};\ \text{AC: } y = \frac{x+5}{2}$$

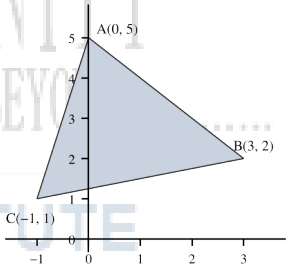
$$\text{Area} = \int_{-1}^{1}\left[\frac{3x+7}{2} - \frac{x+5}{2}\right]dx + \int_1^3\left[\frac{11-x}{2} - \frac{x+5}{2}\right]dx$$

$$= \int_{-1}^{1}(x+1)dx + \int_1^3(3-x)dx = 2+2$$

$\therefore$ Required area = 4 sq. units.

---

9. Using integration, find the area of the triangle ABC whose sides are given by 4x − y + 5 = 0, x + y − 5 = 0 and x − 4y + 5 = 0.



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Solution : Vertices (solving in pairs): A(0, 5), B(3, 2), C(−1, 1).

$$\text{Lines: AC: } y = 4x+5;\ \text{AB: } y = 5-x;\ \text{CB: } y = \frac{x+5}{4}$$

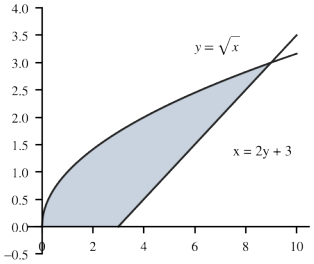
$$\text{Area} = \int_{-1}^{0}\left[(4x+5) - \frac{x+5}{4}\right]dx + \int_0^3\left[(5-x) - \frac{x+5}{4}\right]dx$$

$$= \int_{-1}^{0}\frac{15(x+1)}{4}dx + \int_0^3\frac{5(3-x)}{4}dx = \frac{15}{8} + \frac{45}{8}$$

$\therefore$ Required area $= \frac{15}{2}$ sq. units.

---

10. Find the area of the region bounded by the curve $y = \sqrt{x}$, the line $x = 2y + 3$ and the x-axis, using integration.



Solution : Intersection: $\sqrt{x} = \frac{x-3}{2} \Rightarrow x = 9, y = 3$. Integrate along the y-axis ($0 \le y \le 3$), with the line on the right and the parabola $x = y^2$ on the left:

$$\text{Area} = \int_0^3 [(2y+3) - y^2]dy = \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3$$

$= 9 + 9 - 9$

$\therefore$ Required area = 9 sq. units.

---

11. Find the area of the region enclosed by the curve $y^2 = x$, the lines $x = 1$, $x = 4$ and the x-axis (above the x-axis), using integration.

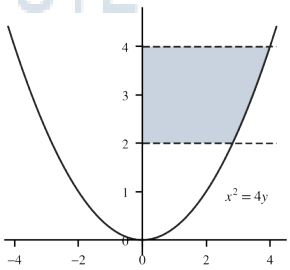
Solution : For the part above the x-axis, $y = \sqrt{x}$.

$$\text{Area} = \int_1^4 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1)$$

$\therefore$ Required area = $\frac{14}{3}$ sq. units.

---

12. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.



Solution : In the first quadrant, $x = 2\sqrt{y}$. Integrate with respect to y:

$$\text{Area} = \int_2^4 2\sqrt{y}\, dy = \left[\frac{4}{3}y^{3/2}\right]_2^4 = \frac{4}{3}(8 - 2\sqrt{2})$$

$\therefore$ Required area = $\frac{32 - 8\sqrt{2}}{3}$ sq. units.

---

13. Find the area enclosed between the parabola $y^2 = 4ax$ and its latus rectum.

Solution : The latus rectum is the line $x = a$. The region is symmetric about the x-axis:

$$\text{Area} = 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx = 4\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a = 4\sqrt{a}\cdot\frac{2}{3}a^{3/2}$$

$\therefore$ Required area $= \frac{8a^2}{3}$ sq. units.

---

14. A farmer has a field bounded by $x = y^2$ and $x = 3$. He wants to divide it equally between his two sons by a straight line $x = c$. Find c.

Solution : Area of the region between $x = y^2$ and $x = k$ (both sides of the x-axis):

$$A(k) = 2\int_0^k \sqrt{x}\,dx = \frac{4}{3}k^{3/2}$$

The line $x = c$ must cut off half of the whole field:

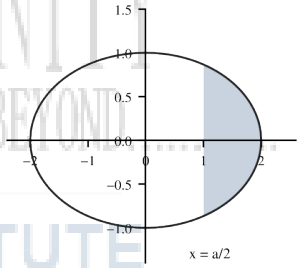
$$\frac{4}{3}c^{3/2} = \frac{1}{2}\cdot\frac{4}{3}\cdot 3^{3/2} \Rightarrow c^{3/2} = \frac{3\sqrt{3}}{2}$$

$$c^3 = \frac{27}{4}$$

$\therefore c = \left(\frac{27}{4}\right)^{1/3} = \frac{3}{\sqrt[3]{4}}$.

---

15. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is divided into two parts by the line $2x = a$. Find the area of the smaller part.



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Solution : The smaller part lies to the right of $x = \frac{a}{2}$ and is symmetric about the x-axis:

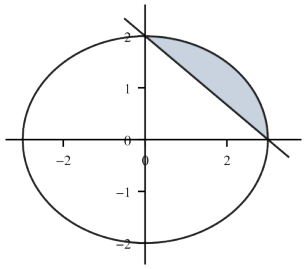
$$\text{Area} = 2\cdot\frac{b}{a}\int_{a/2}^a \sqrt{a^2 - x^2}\,dx = \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/2}^a$$

$$= \frac{2b}{a}\left[\frac{\pi a^2}{4} - \frac{\sqrt{3}a^2}{8} - \frac{\pi a^2}{12}\right] = \frac{2b}{a}\cdot a^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

$\therefore$ Area of the smaller part $= ab\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$ sq. units.

---

16. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.



Solution : The line meets the ellipse at (3, 0) and (0, 2). The smaller region lies between the arc and the chord in the first quadrant:

$$\text{Area} = \int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx - \int_0^3 \frac{2}{3}(3-x)dx$$

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{9\pi}{4} \Rightarrow \text{first integral} = \frac{3\pi}{2}$$

$$\text{Second integral} = \frac{2}{3}\cdot\frac{9}{2} = 3$$

$\therefore$ Required area $= \frac{3}{2}(\pi-2)$ sq. units.

---

17. Find the area of the ellipse $4x^2+9y^2=36$ using integration.

Solution : Standard form: $\frac{x^2}{9}+\frac{y^2}{4}=1$, so a = 3, b = 2 and $y=\frac{2}{3}\sqrt{9-x^2}$.

By symmetry, area = 4 × area in the first quadrant:

$$\text{Area} = 4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx = \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{8}{3}\cdot\frac{9\pi}{4}$$

$\therefore$ Area = 6π sq. units.

---

18. Find the area bounded by the curve y = sin x between x = 0 and x = 2π.

Solution : $\sin x \geq 0$ on $[0, \pi]$ and ≤ 0 on $[\pi, 2\pi]$:

$$\text{Area} = \int_0^{\pi} \sin x\,dx + \left|\int_{\pi}^{2\pi} \sin x\,dx\right|$$

$$= [-\cos x]_0^{\pi} + |[-\cos x]_{\pi}^{2\pi}| = 2 + |-2|$$

$\therefore$ Required area = 4 sq. units.

---

19. Using integration, find the area bounded by the curve |x| + |y| = 1.

Solution : The curve is a square with vertices (±1, 0), (0, ±1), symmetric in all four quadrants. In the first quadrant it is x + y = 1:

$$\text{Area} = 4\int_0^1 (1-x)dx = 4\left[x-\frac{x^2}{2}\right]_0^1 = 4\times\frac{1}{2}$$

$\therefore$ Required area = 2 sq. units.

---

20. Find the area of the region bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$.

Solution : $x^3 < 0$ on $[-2, 0)$ and > 0 on $(0, 1]$:

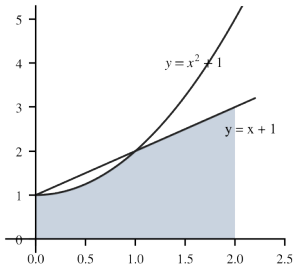
$$\text{Area} = \left|\int_{-2}^{0} x^3 dx\right| + \int_{0}^{1} x^3 dx = \left|\left[\frac{x^4}{4}\right]_{-2}^{0}\right| + \frac{1}{4}$$

$$= 4 + \frac{1}{4}$$

$\therefore$ Required area $= \frac{17}{4}$ sq. units.

---

21. Make a rough sketch of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$ and find its area using integration.



Solution : $x^2 + 1 \le x + 1$ for $0 \le x \le 1$, and $x + 1 \le x^2 + 1$ for $1 \le x \le 2$. So the upper boundary is the smaller of the two:

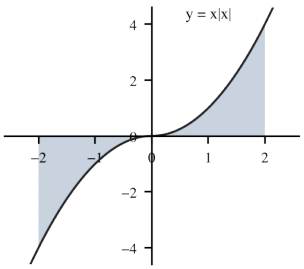
$$\text{Area} = \int_0^1 (x^2 + 1)dx + \int_1^2 (x + 1)dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2 = \frac{4}{3} + \frac{5}{2}$$

$\therefore$ Required area $= \frac{23}{6}$ sq. units.

---

22. Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, the x-axis and the ordinates $x = -2$ and $x = 2$, using integration.



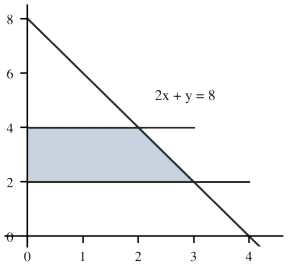
Solution : $y = x^2$ for $x \ge 0$ and $y = -x^2$ for $x < 0$ (below the x-axis). By symmetry:

$$\text{Area} = \left|\int_{-2}^{0} -x^2 dx\right| + \int_{0}^{2} x^2 dx = \frac{8}{3} + \frac{8}{3}$$

$\therefore$ Area $= \frac{16}{3}$ sq units.

---

23. Sketch the region bounded by the lines 2x + y = 8, y = 2, y = 4 and the y-axis. Hence, obtain its area using integration.



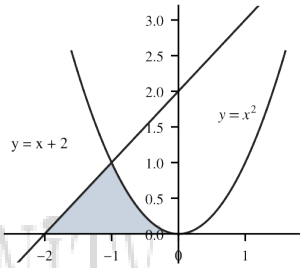
Solution : Using horizontal strips, $x = \frac{8-y}{2}$ from y = 2 to y = 4:

$$\text{Area} = \int_2^4 \frac{8-y}{2}dy = \left[4y - \frac{y^2}{4}\right]_2^4 = (16-4) - (8-1)$$

∴ Area = 5 sq units.

---

24. Find the area of the region bounded by the curves $x^2 = y$, y = x + 2 and the x-axis, using integration.



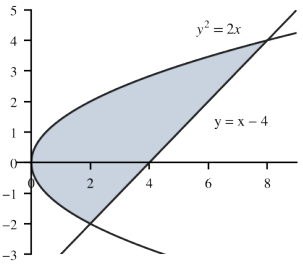
Solution : The line meets the x-axis at (−2, 0) and the parabola at (−1, 1) (in the region shown). The region is bounded above by the line on [−2, −1] and by the parabola on [−1, 0]:

$$\text{Area} = \int_{-2}^{-1}(x+2)dx + \int_{-1}^{0}x^2dx = \frac{1}{2} + \frac{1}{3}$$

∴ Area = $\frac{5}{6}$ sq units.

---

25. Find the area of the region $\{(x, y) : y^2 \le 2x$ and $y \ge x - 4\}$ using integration.



Solution : Points of intersection: $\frac{y^2}{2} = y + 4 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow y = 4, -2$.

Using horizontal strips (line on the right, parabola on the left):

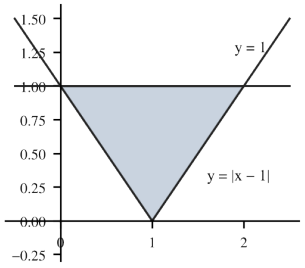
$$\text{Area} = \int_{-2}^{4}\left(y + 4 - \frac{y^2}{2}\right)dy = \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$

$$= \frac{40}{3} - \left(-\frac{14}{3}\right)$$

∴ Area = 18 sq units.

---

26. Find the area bounded by the curves y = |x − 1| and y = 1, using integration.



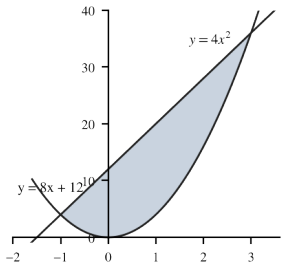
Solution : They meet at x = 0 and x = 2; on [0, 2] the line y = 1 is above:

$$\text{Area} = \int_0^1 [1-(1-x)]dx + \int_1^2 [1-(x-1)]dx = \frac{1}{2} + \frac{1}{2}$$

∴ Area = 1 sq unit.

---

27. Find the area of the region bounded by the curve $4x^2$ = y and the line y = 8x + 12, using integration.



Solution : Intersection: $4x^2 = 8x + 12 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1, 3$. The line is above the parabola:

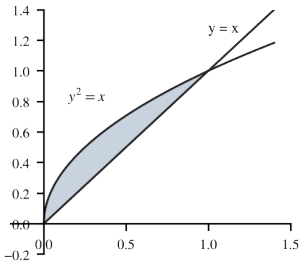
$$\text{Area} = \int_{-1}^{3} (8x + 12 - 4x^2)dx = \left[4x^2 + 12x - \frac{4x^3}{3}\right]_{-1}^{3}$$

$$= 36 - \left(-\frac{20}{3}\right)$$

∴ Area = $\frac{128}{3}$ sq units.

---

28. Using integration, find the area of the region {(x, y) : $y^2 \le x \le y$}.



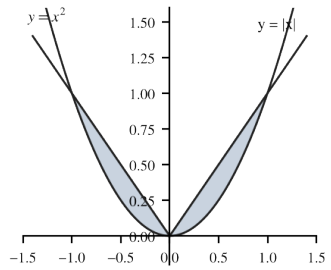
Solution : $y^2 \le y$ gives $0 \le y \le 1$. For each such y, x runs from y^2 to y:

$$\text{Area} = \int_0^1 (y - y^2)dy = \frac{1}{2} - \frac{1}{3}$$

∴ Area = $\frac{1}{6}$ sq units.

---

29. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.



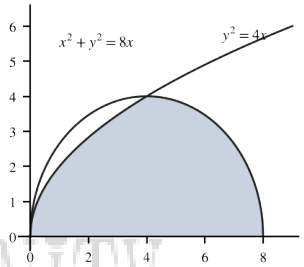
Solution : They meet at (−1, 1), (0, 0), (1, 1); $|x| \geq x^2$ on [−1, 1]. By symmetry:

$$\text{Area} = 2\int_0^1 (x - x^2)dx = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$\therefore$ Area $= \frac{1}{3}$ sq unit.

---

30. Find the area of the region lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$.



Solution : Intersection: $x^2 + 4x = 8x \Rightarrow x = 0$ or 4. The circle has centre (4, 0) and radius 4. On [0, 4] the parabola is the upper boundary; on [4, 8] the circle:

$$\text{Area} = \int_0^4 2\sqrt{x}\,dx + \int_4^8 \sqrt{16 - (x-4)^2}\,dx = \frac{32}{3} + \frac{1}{4}\pi(4)^2$$

$\therefore$ Area $= \left(\frac{32}{3} + 4\pi\right) = \frac{4}{3}(8 + 3\pi)$ sq units.

Chapter 9 – Differential Equations

1. Find the order and degree (if defined) of the differential equation $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$.

Solution : The highest order derivative present is $\frac{d^4y}{dx^4}$, so the order is 4.

The equation is not a polynomial in the derivatives, because $\frac{d^3y}{dx^3}$ appears inside sine.

$\therefore$ Order = 4; degree is not defined.

---

2. Solve: $\frac{dy}{dx} = x^3 \operatorname{cosec} y$, given that $y(0) = 0$.

Solution : Separate the variables:

$$\sin y\,dy = x^3 dx$$

Integrate:

$$-\cos y = \frac{x^4}{4} + C$$

Use $y(0) = 0$: $-1 = 0 + C \Rightarrow C = -1$.

$$-\cos y = \frac{x^4}{4} - 1$$

$$\therefore \cos y = 1 - \frac{x^4}{4}.$$

---

3. Solve the differential equation: $y \log y\, dx - x\, dy = 0$.

Solution : Separate the variables:

$$\frac{dy}{y \log y} = \frac{dx}{x}$$

Integrate (put $\log y = t$ on the left):

$$\log|\log y| = \log|x| + \log C \Rightarrow \log y = Cx$$

$$\therefore y = e^{Cx}.$$

---

4. Solve the differential equation: $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$, given $y(0) = 0$.

Solution : Separate the variables:

$$\frac{dy}{2e^{-y} - 1} = \frac{dx}{x + 1} \Rightarrow \frac{e^y dy}{2 - e^y} = \frac{dx}{x + 1}$$

Integrate (put $2 - e^y = t$):

$$-\log|2 - e^y| = \log|x + 1| + \log C \Rightarrow (2 - e^y)(x + 1) = \frac{1}{C} = K$$

Use $y(0) = 0$: $(2 - 1)(1) = K \Rightarrow K = 1$:

$$2 - e^y = \frac{1}{x + 1} \Rightarrow e^y = \frac{2x + 1}{x + 1}$$

$$\therefore y = \log\left|\frac{2x + 1}{x + 1}\right|, \; x \neq -1.$$

---

5. Find the general solution of the differential equation $\log\left(\frac{dy}{dx}\right) = ax + by$.

Solution :

$$\frac{dy}{dx} = e^{ax + by} = e^{ax}e^{by}$$

Separate the variables:

$$e^{-by}dy = e^{ax}dx$$

Integrate:

$$\therefore -\frac{e^{-by}}{b} = \frac{e^{ax}}{a} + C.$$

---

6. Find the particular solution of the differential equation $\frac{dy}{dx} = \sin(x + y) + \sin(x - y)$, given that $y = 0$ when $x = \frac{\pi}{4}$.

Solution : Use $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$:

$$\frac{dy}{dx} = 2 \sin x \cos y \Rightarrow \sec y \, dy = 2 \sin x \, dx$$

Integrate:

$$\log|\sec y + \tan y| = -2 \cos x + C$$

At $x = \frac{\pi}{4}$, $y = 0$: $0 = -\sqrt{2} + C \Rightarrow C = \sqrt{2}$

$$\therefore \log|\sec y + \tan y| = \sqrt{2} - 2 \cos x.$$

---

7. Find the particular solution of $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$, given that $y = \frac{\pi}{2}$ when $x = 1$.

Solution : Separate the variables:

$$(\sin y + y \cos y) \, dy = x(2 \log x + 1) \, dx$$

Integrate: note $\frac{d}{dy}(y \sin y) = \sin y + y \cos y$ and $\frac{d}{dx}(x^2 \log x) = 2x \log x + x$:

$$y \sin y = x^2 \log x + C$$

At $x = 1$, $y = \frac{\pi}{2}$: $\frac{\pi}{2} = 0 + C$

$$\therefore y \sin y = x^2 \log x + \frac{\pi}{2}.$$

---

8. Solve: $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$.

Solution : Factorise: $1 + x^2 + y^2 + x^2y^2 = (1 + x^2)(1 + y^2)$.

$$xy\frac{dy}{dx} = -\sqrt{1 + x^2}\sqrt{1 + y^2} \Rightarrow \frac{y \, dy}{\sqrt{1 + y^2}} = -\frac{\sqrt{1 + x^2}}{x} dx$$

Left side: $\int \frac{y}{\sqrt{1 + y^2}} dy = \sqrt{1 + y^2}$.

Right side: put $1 + x^2 = t^2$, $x \, dx = t \, dt$:

$$\int\frac{\sqrt{1+x^2}}{x}dx = \int\frac{t^2}{t^2-1}dt = t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$\therefore \sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = C.$$

---

9. Prove that $x^2 - y^2 = C(x^2+y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where C is a parameter.

Solution : Differentiate $x^2 - y^2 = C(x^2+y^2)^2$ with respect to x:

$$2x - 2y\frac{dy}{dx} = 2C(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

Eliminate C using $C = \frac{x^2-y^2}{(x^2+y^2)^2}$:

$$\left(x - y\frac{dy}{dx}\right)(x^2+y^2) = 2(x^2-y^2)\left(x + y\frac{dy}{dx}\right)$$

$$x^3 + xy^2 - 2x^3 + 2xy^2 = y\frac{dy}{dx}[(x^2+y^2) + 2(x^2-y^2)]$$

$$-(x^3 - 3xy^2) = y(3x^2 - y^2)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$$

This is the given equation, and the family has one arbitrary constant (first-order equation).

$\therefore x^2 - y^2 = C(x^2+y^2)^2$ is its general solution.

---

10. Show that the differential equation $(x - y)\dfrac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution :

$\frac{dy}{dx} = \frac{x+2y}{x-y} = F(x, y)$, and $F(\lambda x, \lambda y) = F(x, y)$, so it is homogeneous

Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\frac{1-v}{v^2+v+1}dv = \frac{dx}{x}$$

Write $1 - v = -\frac{1}{2}(2v+1) + \frac{3}{2}$:

$$-\frac{1}{2}\log(v^2+v+1) + \sqrt{3}\tan^{-1}\frac{2v+1}{\sqrt{3}} = \log|x| + C$$

Put $v = y/x$ and simplify the logarithms:

$$\therefore -\frac{1}{2}\log|x^2+xy+y^2| + \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) = C.$$

11. Solve the differential equation: $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$.

Solution :

$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ — homogeneous. Put $y = vx$:

$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrate:

$\log|v+\sqrt{1+v^2}| = \log|x| + \log C \Rightarrow v + \sqrt{1+v^2} = Cx$

Put $v = \frac{y}{x}$:

$\therefore y + \sqrt{x^2+y^2} = Cx^2$.

12. Solve the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$. Find the particular solution given that $y = \frac{\pi}{2}$ when $x = 1$.

Solution : It is homogeneous. Put $y = vx$:

$x\left(v + x\frac{dv}{dx}\right)\sin v + x - vx\sin v = 0 \Rightarrow x^2\sin v\frac{dv}{dx} = -x$

$\sin v\,dv = -\frac{dx}{x} \Rightarrow -\cos v = -\log|x| + C$

So $\cos\left(\frac{y}{x}\right) = \log|x| + C_1$. At $x = 1$, $y = \pi/2$: $\cos\frac{\pi}{2} = 0 = 0 + C_1 \Rightarrow C_1 = 0$.

$\therefore \cos\left(\frac{y}{x}\right) = \log|x|$.

13. Solve: $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x,\ x \neq 0$.

Solution : Homogeneous; put $y = vx$:

$x\cos v\left(v + x\frac{dv}{dx}\right) = vx\cos v + x \Rightarrow x\cos v\frac{dv}{dx} = 1$

$\cos v\,dv = \frac{dx}{x} \Rightarrow \sin v = \log|x| + C$

$\therefore \sin\left(\frac{y}{x}\right) = \log|x| + C$.

14. Solve the differential equation: $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

Solution : Write it as $\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}}$ — homogeneous in x/y. Put $x = vy$, $\frac{dx}{dy} = v + y\frac{dv}{dy}$:

$$v + y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} \Rightarrow y\frac{dv}{dy} = -\frac{v + e^v}{1 + e^v}$$

$$\frac{1 + e^v}{v + e^v}dv = -\frac{dy}{y} \Rightarrow \log|v + e^v| = -\log|y| + \log C$$

$$y(v + e^v) = C$$

$\therefore x + ye^{x/y} = C.$

---

15. Solve: $ye^{x/y}dx = (xe^{x/y} + y^2)dy,\ y \neq 0.$

Solution :

$\frac{dx}{dy} = \frac{x}{y} + ye^{-x/y}$. Put $x = vy$:

$$v + y\frac{dv}{dy} = v + ye^{-v} \Rightarrow e^v dv = dy$$

$$e^v = y + C$$

$\therefore e^{x/y} = y + C.$

---

16. Solve the differential equation: $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1},\ |x| > 1.$

Solution : Divide by $x^2 - 1$: linear in y with $P = \frac{2x}{x^2 - 1}$, $Q = \frac{2}{(x^2 - 1)^2}$.

$$\text{I.F.} = e^{\int \frac{2x}{x^2 - 1}dx} = x^2 - 1$$

$$y(x^2 - 1) = \int \frac{2}{x^2 - 1}dx = \log\left|\frac{x - 1}{x + 1}\right| + C$$

$\therefore y(x^2 - 1) = \log\left|\frac{x - 1}{x + 1}\right| + C.$

---

17. Find the particular solution of $(1 + x^3)\frac{dy}{dx} + 6x^2y = 1 + x^2$, given that y = 1 when x = 1.

Solution : Linear: $\frac{dy}{dx} + \frac{6x^2}{1 + x^3}y = \frac{1 + x^2}{1 + x^3}$

$$\text{I.F.} = e^{2\log(1 + x^3)} = (1 + x^3)^2$$

$$y(1 + x^3)^2 = \int (1 + x^2)(1 + x^3)dx = \int (1 + x^2 + x^3 + x^5)dx$$

$$= x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$$

At $x = 1, y = 1: 4 = 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + C \Rightarrow C = \frac{9}{4}$

$\therefore y(1 + x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{9}{4}.$

---

18. Find the general solution of $(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$.

Solution : The left side is $\frac{d}{dx}[y(x^2 + 1)]$ (I.F. $= x^2 + 1$):

$y(x^2 + 1) = \int \sqrt{x^2 + 4}\, dx$

$= \frac{x}{2}\sqrt{x^2 + 4} + 2\log|x + \sqrt{x^2 + 4}| + C$

$\therefore y(x^2 + 1) = \frac{x}{2}\sqrt{x^2 + 4} + 2\log|x + \sqrt{x^2 + 4}| + C.$

---

19. Solve: $\cos^2 x\frac{dy}{dx} + y = \tan x$, $0 \le x < \frac{\pi}{2}$.

Solution :

$\frac{dy}{dx} + \sec^2 x \cdot y = \tan x \sec^2 x$; I.F. $= e^{\tan x}$

$ye^{\tan x} = \int \tan x \sec^2 x\, e^{\tan x} dx$

Put $\tan x = t$: $\int te^t dt = (t - 1)e^t$

$ye^{\tan x} = (\tan x - 1)e^{\tan x} + C$

$\therefore y = \tan x - 1 + Ce^{-\tan x}.$

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20. Find the general solution of $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$.

Solution : Write it as a linear equation in x:

$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{\tan^{-1} y}}{1 + y^2}$; I.F. $= e^{\tan^{-1} y}$

$xe^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1 + y^2} dy$

Put $\tan^{-1} y = t$:

$\int e^{2t} dt = \frac{1}{2}e^{2t}$

$\therefore xe^{\tan^{-1} y} = \frac{1}{2}e^{2\tan^{-1} y} + C.$

---

21. Solve: $y\,dx + (x - y^2)\,dy = 0$.

Solution : Linear in x:

$$\frac{dx}{dy} + \frac{x}{y} = y;\ \text{I.F.} = e^{\log y} = y$$

$$xy = \int y^2 dy = \frac{y^3}{3} + C$$

$$\therefore xy = \frac{y^3}{3} + C.$$

---

22. Find the particular solution of $x\frac{dy}{dx} + y + \frac{1}{1+x^2} = 0$, given that y(1) = 0.

Solution : The first two terms form $\frac{d}{dx}(xy)$:

$$\frac{d}{dx}(xy) = -\frac{1}{1+x^2} \Rightarrow xy = -\tan^{-1} x + C$$

At x = 1, y = 0: $0 = -\frac{\pi}{4} + C \Rightarrow C = \frac{\pi}{4}$

$$\therefore xy = \frac{\pi}{4} - \tan^{-1} x.$$

---

23. Find the particular solution of $x\frac{dy}{dx} + y = x\cos x + \sin x$, given $y\left(\frac{\pi}{2}\right) = 1$.

Solution :

$$\frac{d}{dx}(xy) = x\cos x + \sin x = \frac{d}{dx}(x\sin x)$$

$$xy = x\sin x + C$$

At $x = \frac{\pi}{2}$, y = 1: $\frac{\pi}{2} = \frac{\pi}{2} + C \Rightarrow C = 0$

$$\therefore y = \sin x.$$

---

24. Find the particular solution of $\frac{dy}{dx} + y\cot x = 2x + x^2\cot x$ ($x \neq 0$), given that y = 0 when $x = \frac{\pi}{2}$

.

Solution :

$$\text{I.F.} = e^{\int \cot x\, dx} = \sin x$$

$$y\sin x = \int (2x\sin x + x^2\cos x)dx = x^2\sin x + C \ (\text{since } \frac{d}{dx}(x^2\sin x) = 2x\sin x + x^2\cos x)$$

At $x = \frac{\pi}{2}$, y = 0: $0 = \frac{\pi^2}{4} + C \Rightarrow C = -\frac{\pi^2}{4}$

$$\therefore y = x^2 - \frac{\pi^2}{4\sin x}\ (\sin x \neq 0).$$

Chapter 10 – Vector Algebra

1. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, find a unit vector along $\vec{a} \times \vec{b}$.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \hat{i}(-1+2) - \hat{j}(4-2) + \hat{k}(-8+2) = \hat{i} - 2\hat{j} - 6\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{1+4+36} = \sqrt{41}$$

$\therefore$ Unit vector $= \frac{1}{\sqrt{41}}(\hat{i} - 2\hat{j} - 6\hat{k})$.

---

2. Find the area of the parallelogram whose adjacent sides are determined by $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution : Area $= |\vec{a} \times \vec{b}|$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{400+25+25} = \sqrt{450}$$

$\therefore$ Area $= 15\sqrt{2}$ sq. units.

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3. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides AB and AC respectively of ΔABC. Find the length of the median through A.

Solution : The median AD goes to the mid-point D of BC:

$$\vec{AD} = \frac{1}{2}(\vec{AB} + \vec{AC}) = \frac{1}{2}(3\hat{i} + 0\hat{j} + 5\hat{k})$$

$$|\vec{AD}| = \frac{1}{2}\sqrt{9+25}$$

$\therefore$ Length of the median $= \frac{\sqrt{34}}{2}$ units.

---

4. If ABCD is a quadrilateral and E, F are the mid-points of AC and BD respectively, prove that $\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\vec{EF}$.

Solution : Let the position vectors of A, B, C, D be $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Then E $= \frac{\vec{a}+\vec{c}}{2}$ and F $= \frac{\vec{b}+\vec{d}}{2}$.

LHS $= (\vec{b} - \vec{a}) + (\vec{d} - \vec{a}) + (\vec{b} - \vec{c}) + (\vec{d} - \vec{c}) = 2(\vec{b} + \vec{d}) - 2(\vec{a} + \vec{c})$

$$4\vec{EF} = 4\left(\frac{\vec{b}+\vec{d}}{2} - \frac{\vec{a}+\vec{c}}{2}\right) = 2(\vec{b}+\vec{d}) - 2(\vec{a}+\vec{c})$$

$\therefore$ LHS = RHS.

---

5. If $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$, find the unit vector in the direction of $\vec{a} + \vec{b}$.

Solution :

$\vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$, $|\vec{a} + \vec{b}| = \sqrt{9 + 36 + 4} = 7$

$\therefore$ Unit vector $= \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$.

---

6. Find the position vector of the point which divides the join of the points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ internally in the ratio 2 : 1.

Solution : Section formula (internal, m : n = 2 : 1):

$$\vec{r} = \frac{2(2\vec{a} + 3\vec{b}) + 1(3\vec{a} - 2\vec{b})}{2 + 1} = \frac{7\vec{a} + 4\vec{b}}{3}$$

$\therefore$ Position vector $= \frac{7\vec{a} + 4\vec{b}}{3}$.

---

7. Find the direction cosines of the vector joining the points A(1, 2, −3) and B(−1, −2, 1), directed from B to A.

Solution :

$\vec{BA} = (1 + 1)\hat{i} + (2 + 2)\hat{j} + (-3 - 1)\hat{k} = 2\hat{i} + 4\hat{j} - 4\hat{k}$

$|\vec{BA}| = \sqrt{4 + 16 + 16} = 6$

$\therefore$ Direction cosines $= \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$.

---

8. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

Solution : Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$.

$\vec{a} \times \vec{c} = (z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k} = \hat{j} - \hat{k}$

$z - y = 0$, $x - z = 1$, $y - x = -1$

$\vec{a} \cdot \vec{c} = x + y + z = 3$

With $y = z$ and $x = z + 1$: $(z + 1) + z + z = 3 \Rightarrow z = \frac{2}{3}$, $y = \frac{2}{3}$, $x = \frac{5}{3}$

$\therefore \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$.

---

9. If the vector $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between the vector $\vec{c}$ and the vector $3\hat{i}+4\hat{j}$, find the unit vector in the direction of $\vec{c}$.

Solution : The internal bisector is along the sum of the unit vectors: $\hat{c}+\dfrac{3\hat{i}+4\hat{j}}{5}=\lambda(-\hat{i}+\hat{j}-\hat{k})$.

$$\hat{c}=\left(-\lambda-\frac{3}{5}\right)\hat{i}+\left(\lambda-\frac{4}{5}\right)\hat{j}-\lambda\hat{k}$$

$|\hat{c}| = 1$:

$$\left(\lambda+\frac{3}{5}\right)^2+\left(\lambda-\frac{4}{5}\right)^2+\lambda^2=1\Rightarrow 3\lambda^2-\frac{2}{5}\lambda=0$$

$\lambda = 0$ would make $\hat{c}$ opposite to the given vector, so $\lambda=\dfrac{2}{15}$:

$$\therefore \hat{c}=-\frac{1}{15}(11\hat{i}+10\hat{j}+2\hat{k}).$$

---

10. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$, and $\vec{d}\times\vec{a}=21$.

Solution : $\vec{d}$ is parallel to $\vec{c}\times\vec{b}$:

$$\vec{c}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&-1\\1&-4&5\end{vmatrix}=\hat{i}(5-4)-\hat{j}(15+1)+\hat{k}(-12-1)=\hat{i}-16\hat{j}-13\hat{k}$$

$$\vec{d}=\lambda(\hat{i}-16\hat{j}-13\hat{k})$$

$$\vec{d}\cdot\vec{a}=\lambda(4-80+13)=-63\lambda=21\Rightarrow\lambda=-\frac{1}{3}$$

$$\therefore \vec{d}=\frac{1}{3}(-\hat{i}+16\hat{j}+13\hat{k}).$$

---

11. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}-\vec{c}=\vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution :

$$\vec{a}+\vec{b}=\vec{c}\Rightarrow|\vec{a}+\vec{b}|^2=|\vec{c}|^2$$

$$9+25+2\vec{a}\cdot\vec{b}=49\Rightarrow\vec{a}\cdot\vec{b}=\frac{15}{2}$$

$$\cos\theta=\frac{15/2}{3\times5}=\frac{1}{2}$$

$$\therefore \text{Angle}=\frac{\pi}{3}\ (60^\circ).$$

---

12. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

Solution :

$\vec{d_1}=\vec{a}+\vec{b}=4\hat{i}-2\hat{j}-2\hat{k}$, $|\vec{d_1}|=2\sqrt{6}$

$\vec{d_2}=\vec{a}-\vec{b}=-6\hat{j}-8\hat{k}$, $|\vec{d_2}|=10$

Unit vectors: $\frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$ and $-\frac{1}{5}(3\hat{j}+4\hat{k})$.

Area $=\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$:

$$\vec{d_1}\times\vec{d_2}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&-2&-2\\0&-6&-8\end{vmatrix}=4\hat{i}+32\hat{j}-24\hat{k},\ \text{magnitude}\ \sqrt{1616}=4\sqrt{101}$$

$\therefore$ Unit vectors as above; area $=2\sqrt{101}$ sq. units.

---

13. Find the value of λ for which the scalar projection of $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units.

Solution :

$$\text{Projection}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{2\lambda+6+12}{7}=4$$

$2\lambda+18=28$

$\therefore \lambda=5$.

---

14. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, prove that $2\vec{a}+\vec{b}$ is perpendicular to $\vec{b}$.

Solution : Square both sides:

$|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{a}|^2\Rightarrow 2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=0$

$(2\vec{a}+\vec{b})\cdot\vec{b}=0$

$\therefore 2\vec{a}+\vec{b}$ is perpendicular to $\vec{b}$.

---

15. If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, find λ so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

Solution :

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2=0$

$(1+1+49)-(25+1+\lambda^2)=0\Rightarrow\lambda^2=25$

$\therefore \lambda=\pm5$.

---

16. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, find λ.

Solution :

$$\vec{a} + \lambda\vec{b} = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$$

$$(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 3(2 - \lambda) + (2 + 2\lambda) = 8 - \lambda = 0$$

$\therefore \lambda = 8.$

---

17. For two non-zero vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}|$, find the angle between them. Also, if $\vec{a}$ and $\vec{b}$ are unit vectors with $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$, find the angle between them.

Solution : Part 1: square both sides:

$$|\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \Rightarrow \vec{a} \cdot \vec{b} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Part 2: square both sides with $|\vec{a}| = |\vec{b}| = 1$:

$$4 + 9 + 12\vec{a} \cdot \vec{b} = 9 + 4 - 12\vec{a} \cdot \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$$

$\therefore$ In both cases the angle is $\frac{\pi}{2}$.

---

18. If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = 400$ and $|\vec{b}| = 5$, find $|\vec{a}|$.

Solution :

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2$$

$$25|\vec{a}|^2 = 400 \Rightarrow |\vec{a}|^2 = 16$$

$\therefore |\vec{a}| = 4.$

---

19. If $\vec{a}$ and $\vec{b}$ are two vectors of equal magnitude and α is the angle between them, prove that $\dfrac{|\vec{a} + \vec{b}|}{|\vec{a} - \vec{b}|} = \cot\dfrac{\alpha}{2}$.

Solution : Let $|\vec{a}| = |\vec{b}| = k$.

$$|\vec{a} + \vec{b}|^2 = 2k^2(1 + \cos\alpha) = 4k^2\cos^2\frac{\alpha}{2}$$

$$|\vec{a} - \vec{b}|^2 = 2k^2(1 - \cos\alpha) = 4k^2\sin^2\frac{\alpha}{2}$$

Divide and take square roots ($0 < \alpha < \pi$):

$$\therefore \frac{|\vec{a} + \vec{b}|}{|\vec{a} - \vec{b}|} = \cot\frac{\alpha}{2}.$$

---

20. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Solution :

$\text{Sum} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$, magnitude $\sqrt{(2+\lambda)^2+40}$

$$\frac{(2+\lambda)+6-2}{\sqrt{(2+\lambda)^2+40}} = 1 \Rightarrow (\lambda+6)^2 = (\lambda+2)^2+40$$

$$12\lambda+36 = 4\lambda+44 \Rightarrow 8\lambda = 8$$

$\therefore \lambda = 1.$

---

21. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitude, show that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also find the angle.

Solution : Let each have magnitude k, with $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$.

$$|\vec{a}+\vec{b}+\vec{c}|^2 = 3k^2 \Rightarrow |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}k$$

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{\sqrt{3}k\cdot k} = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}}$$

By the same working, the angles with $\vec{b}$ and $\vec{c}$ also have cosine $\frac{1}{\sqrt{3}}$.

$\therefore$ The vector is equally inclined to all three, at an angle of $\cos^{-1}\frac{1}{\sqrt{3}}$.

---

22. If $\vec{\alpha} = 3\hat{i}-\hat{j}$ and $\vec{\beta} = 2\hat{i}+\hat{j}-3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta_1}+\vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.

Solution : Let $\vec{\beta_1} = \lambda\vec{\alpha}$; then $\vec{\beta_2} = \vec{\beta}-\lambda\vec{\alpha}$ and $\vec{\beta_2}\cdot\vec{\alpha} = 0$:

$$\vec{\beta}\cdot\vec{\alpha}-\lambda|\vec{\alpha}|^2 = 0 \Rightarrow 5-10\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$$

$$\vec{\beta_1} = \frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}$$

$$\vec{\beta_2} = \vec{\beta}-\vec{\beta_1} = \frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

$$\therefore \vec{\beta} = \left(\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}\right)+\left(\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}\right).$$

---

23. Find the area of the triangle with vertices A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1).

Solution :

$$\vec{AB} = \hat{j}+2\hat{k},\ \vec{AC} = \hat{i}+2\hat{j}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \hat{i}(0-4) - \hat{j}(0-2) + \hat{k}(0-1) = -4\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{16+4+1}$$

$\therefore$ Area $= \dfrac{\sqrt{21}}{2}$ sq. units.

---

24. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear.

Solution :

$$\vec{AB} = 3\hat{i} - \hat{j} - 2\hat{k},\ \vec{BC} = 6\hat{i} - 2\hat{j} - 4\hat{k} = 2\vec{AB}$$

$\vec{BC}$ is parallel to $\vec{AB}$, and B is a common point.

$\therefore$ A, B and C are collinear.

Chapter 11 – Three Dimensional Geometry

1. The Cartesian equation of a line AB is $\dfrac{2x-1}{12} = \dfrac{y+2}{2} = \dfrac{z-3}{3}$. Find the direction cosines of a line parallel to AB.

Solution : Write the x-term in standard form (coefficient of x equal to 1):

$$\frac{x-\frac{1}{2}}{6} = \frac{y+2}{2} = \frac{z-3}{3} \Rightarrow \text{direction ratios } 6, 2, 3$$

$$\sqrt{36+4+9} = 7$$

$\therefore$ Direction cosines $= \dfrac{6}{7}, \dfrac{2}{7}, \dfrac{3}{7}$ (or their negatives).

---

2. Find the direction cosines of the line $\dfrac{x+2}{2} = \dfrac{2y-7}{6} = \dfrac{5-z}{6}$. Also find the vector equation of the line through A(−1, 2, 3) parallel to it.

Solution : Standard form:

$$\frac{x+2}{2} = \frac{y-\frac{7}{2}}{3} = \frac{z-5}{-6} \Rightarrow \text{direction ratios } 2, 3, -6;\ \sqrt{4+9+36} = 7$$

Direction cosines: $\dfrac{2}{7}, \dfrac{3}{7}, -\dfrac{6}{7}$

Line through A parallel to it:

$\therefore \vec{r} = (-\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$.

---

3. Write the direction ratios of the vector $3\vec{a} + 2\vec{b}$, where $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$.

Solution :

$3\vec{a} + 2\vec{b} = (3 + 4)\hat{i} + (3 - 8)\hat{j} + (-6 + 10)\hat{k} = 7\hat{i} - 5\hat{j} + 4\hat{k}$

$\therefore$ Direction ratios: 7, −5, 4.

---

4. Find the Cartesian and vector equations of the line through the points (2, −3, 1) and (3, −4, −5).

Solution :

Direction ratios: 3 − 2, −4 + 3, −5 − 1 = 1, −1, −6

Cartesian:

$$\frac{x - 2}{1} = \frac{y + 3}{-1} = \frac{z - 1}{-6}$$

Vector:

$\therefore \vec{r} = (2\hat{i} - 3\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} - 6\hat{k})$; Cartesian form as above.

---

5. Find the equation of the line passing through (2, 0, 5) and parallel to the line $6x - 2 = 3y + 1 = 2z - 2$.

Solution : Divide the given equation by 6 to bring it to standard form:

$$\frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z - 1}{3} \Rightarrow \text{direction ratios } 1, 2, 3$$

$\therefore$ Required line: $\dfrac{x - 2}{1} = \dfrac{y}{2} = \dfrac{z - 5}{3}$.

---

6. (i) Find the vector and Cartesian equations of the line through (−5, 7, −4) in the direction (3, −2, 1). (ii) Find the point where this line crosses the xy-plane.

Solution : (i)

$$\vec{r} = (-5\hat{i} + 7\hat{j} - 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + \hat{k}); \quad \frac{x + 5}{3} = \frac{y - 7}{-2} = \frac{z + 4}{1}$$

(ii) A general point is $(-5 + 3\lambda, 7 - 2\lambda, -4 + \lambda)$. On the xy-plane z = 0:

$-4 + \lambda = 0 \Rightarrow \lambda = 4 \Rightarrow (7, -1, 0)$

$\therefore$ The line crosses the xy-plane at (7, −1, 0).

---

7. Find the coordinates of the point where the line through (−1, 1, −8) and (5, −2, 10) crosses the ZX-plane.

Solution :

Direction ratios: 6, −3, 18; general point $(-1 + 6t, 1 - 3t, -8 + 18t)$

On the ZX-plane y = 0:

$$1 - 3t = 0 \Rightarrow t = \frac{1}{3}$$

$\therefore$ Point = (1, 0, –2).

---

8. Find the angle between the lines $\vec{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Solution :

$\vec{b_1} = \hat{i} + 2\hat{j} + 2\hat{k}$ ($|\cdot| = 3$), $\vec{b_2} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ ($|\cdot| = 7$)

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} = \frac{2 + 6 + 12}{21} = \frac{20}{21}$$

$\therefore \theta = \cos^{-1}\frac{20}{21}$.

---

9. Find the value of p so that the lines l_1: $\frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and l_2: $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular. Also find the equation of the line through (3, 2, –4) parallel to l_1.

Solution : Standard forms:

$$l_1: \frac{x-1}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2}; \; l_2: \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$$

Perpendicular $\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$:

$$\frac{9p}{7} + \frac{p}{7} - 10 = 0 \Rightarrow p = 7$$

Then l_1 has direction ratios –3, 1, 2:

$\therefore$ p = 7; required line $\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2}$.

---

10. Find the equation of a line passing through the point (1, 2, –4) and perpendicular to the lines $\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

Solution : The required direction is along $\vec{b_1} \times \vec{b_2}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48) = 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Parallel to $2\hat{i} + 3\hat{j} + 6\hat{k}$

$\therefore \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

---

11. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also find their point of intersection.

Solution : General points: $(3\lambda - 1, 5\lambda - 3, 7\lambda - 5)$ and $(\mu + 2, 3\mu + 4, 5\mu + 6)$. Equate:

$3\lambda - 1 = \mu + 2$ and $5\lambda - 3 = 3\mu + 4 \Rightarrow \lambda = \frac{1}{2}, \mu = -\frac{3}{2}$

Check the z-coordinates:

$7\lambda - 5 = -\frac{3}{2}$, $5\mu + 6 = -\frac{3}{2}$ ✓

So the lines intersect.

$\therefore$ Point of intersection $= \left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$.

---

12. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect. Also find their point of intersection.

Solution : General points: $(1 + 3\lambda, 1 - \lambda, -1)$ and $(4 + 2\mu, 0, -1 + 3\mu)$. Equate:

y: $1 - \lambda = 0 \Rightarrow \lambda = 1$; z: $-1 = -1 + 3\mu \Rightarrow \mu = 0$

x: $1 + 3 = 4 + 0$ ✓

$\therefore$ The lines intersect at $(4, 0, -1)$.

---

13. Prove that the line through A(0, −1, −1) and B(4, 5, 1) intersects the line through C(3, 9, 4) and D(−4, 4, 4).

Solution :

AB: $(4t, -1 + 6t, -1 + 2t)$; CD: $(3 - 7s, 9 - 5s, 4)$

Equate z: $-1 + 2t = 4 \Rightarrow t = \frac{5}{2}$, giving the point $(10, 14, 4)$ on AB.

Equate x: $3 - 7s = 10 \Rightarrow s = -1$, giving $y = 9 + 5 = 14$ on CD ✓

All three coordinates agree.

$\therefore$ The lines intersect at $(10, 14, 4)$.

---

14. Find the shortest distance between the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$.

Solution :

$\vec{a_1} = (3, 5, 7)$, $\vec{b_1} = (1, -2, 1)$; $\vec{a_2} = (-1, -1, -1)$, $\vec{b_2} = (7, -6, 1)$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix} = 4\hat{i} + 6\hat{j} + 8\hat{k}, \text{ magnitude } \sqrt{116} = 2\sqrt{29}$$

$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-4)(4) + (-6)(6) + (-8)(8) = -116$

$\text{SD} = \dfrac{|-116|}{2\sqrt{29}} = \dfrac{58}{\sqrt{29}}$

$\therefore$ Shortest distance $= 2\sqrt{29}$ units.

---

15. An insect is crawling along the line $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and another insect along $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$. At what points on the lines should they reach so that the distance between them is the shortest? Find this shortest distance.

Solution : Let P$(6 + \lambda, 2 - 2\lambda, 2 + 2\lambda)$ and Q$(-4 + 3\mu, -2\mu, -1 - 2\mu)$. PQ must be perpendicular to both lines:

$\vec{PQ} = (-10 + 3\mu - \lambda, -2 - 2\mu + 2\lambda, -3 - 2\mu - 2\lambda)$

$\vec{PQ} \cdot (1, -2, 2) = 0 \Rightarrow (-10 + 3\mu - \lambda) - 2(-2 - 2\mu + 2\lambda) + 2(-3 - 2\mu - 2\lambda) = -12 + 3\mu - 9\lambda = 0$

$\vec{PQ} \cdot (3, -2, -2) = 0 \Rightarrow 3(-10 + 3\mu - \lambda) - 2(-2 - 2\mu + 2\lambda) - 2(-3 - 2\mu - 2\lambda) = -20 + 17\mu - 3\lambda = 0$

Solve: $\lambda = -1$, $\mu = 1$

P $= (5, 4, 0)$, Q $= (-1, -2, -3)$

$PQ = \sqrt{36 + 36 + 9} = 9$

$\therefore$ The insects should be at (5, 4, 0) and (−1, −2, −3); shortest distance = 9 units.

---

16. Find the vector equation of the line passing through (2, 1, −1) and parallel to the line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$. Also find the distance between these two lines.

Solution :

Required line: $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(2\hat{i} - \hat{j} + \hat{k})$

For parallel lines, $d = \dfrac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|}$ with $\vec{a_2} - \vec{a_1} = \hat{i} - \hat{k}$:

$$\vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i}(1 - 0) - \hat{j}(-2 - 1) + \hat{k}(0 + 1) = \hat{i} + 3\hat{j} + \hat{k}$$

$d = \dfrac{\sqrt{11}}{\sqrt{6}}$

$\therefore$ Distance $= \sqrt{\dfrac{11}{6}}$ units.

---

17. Find the coordinates of the foot of the perpendicular and the length of the perpendicular drawn from P(5, 4, 2) to the line $\vec{r} = (-\hat{i} + 3\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also find the image of P in this line.

Solution : General point of the line: Q(−1 + 2λ, 3 + 3λ, 1 − λ).

$\vec{PQ}$ = (−6 + 2λ, −1 + 3λ, −1 − λ)

$\vec{PQ} \perp$ line:

$2(-6 + 2\lambda) + 3(-1 + 3\lambda) - (-1 - \lambda) = 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

Foot Q = (1, 6, 0); PQ = $\sqrt{16 + 4 + 4} = 2\sqrt{6}$

Image P′ satisfies Q = mid-point of PP′:

P′ = 2Q − P = (−3, 8, −2)

∴ Foot (1, 6, 0); length $2\sqrt{6}$ units; image (−3, 8, −2).

---

18. Find the coordinates of the image of the point (1, 6, 3) with respect to the line $\vec{r} = (\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$. Also find the distance of the image from the y-axis.

Solution : General point: Q(λ, 1 + 2λ, 2 + 3λ); $\vec{PQ}$ = (λ − 1, 2λ − 5, 3λ − 1).

$\vec{PQ} \cdot (1, 2, 3) = 14\lambda - 14 = 0 \Rightarrow \lambda = 1 \Rightarrow$ foot Q(1, 3, 5)

Image P′ = 2Q − P = (1, 0, 7)

Distance of (x, y, z) from the y-axis is $\sqrt{x^2 + z^2}$:

$\sqrt{1 + 49} = 5\sqrt{2}$

∴ Image (1, 0, 7); distance from the y-axis = $5\sqrt{2}$ units.

---

19. Find the coordinates of the foot of the perpendicular drawn from the point (3, −1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Hence write the equation of this perpendicular and its length.

Solution : General point: Q(2λ, 2 + 3λ, 3 + 4λ); $\vec{PQ}$ = (2λ − 3, 3 + 3λ, 4λ − 8).

$2(2\lambda - 3) + 3(3 + 3\lambda) + 4(4\lambda - 8) = 29\lambda - 29 = 0 \Rightarrow \lambda = 1$

Foot Q = (2, 5, 7); direction of PQ: (−1, 6, −4)

Length = $\sqrt{1 + 36 + 16} = \sqrt{53}$

∴ Foot (2, 5, 7); perpendicular $\frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}$; length $\sqrt{53}$ units.

---

20. Find the image of the point (2, −1, 5) in the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$. Also find the equation of the line joining the point and its image, and the length of this segment.

Solution : General point: Q(11 + 10λ, −2 − 4λ, −8 − 11λ); $\vec{PQ}$ = (9 + 10λ, −1 − 4λ, −13 − 11λ).

$10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) = 237\lambda + 237 = 0 \Rightarrow \lambda = -1$

Foot Q = (1, 2, 3); image P′ = 2Q − P = (0, 5, 1)

Line PP′: $\frac{x-2}{-2}=\frac{y+1}{6}=\frac{z-5}{-4}$; $PP' = \sqrt{4+36+16} = 2\sqrt{14}$

∴ Image (0, 5, 1); line as above; length $2\sqrt{14}$ units.

---

21. Find the equation of the line which passes through the point (1, 1, 1) and intersects the lines $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

Solution : Let the required line meet the first line at A(−2 + t, 3 + 2t, −1 + 4t) and the second at B(1 + 2u, 2 + 3u, 3 + 4u). Then P(1, 1, 1), A, B are collinear: $\vec{PB} = k\vec{PA}$.

$\vec{PA}$ = (t − 3, 2 + 2t, 4t − 2), $\vec{PB}$ = (2u, 1 + 3u, 2 + 4u)

2u = k(t − 3), 1 + 3u = k(2 + 2t), 2 + 4u = k(4t − 2)

Solving: t = 9, u = $\frac{3}{11}$, k = $\frac{1}{11}$

$\vec{PA}$ = (6, 20, 34), parallel to (3, 10, 17)

∴ Required line: $\frac{x-1}{3}=\frac{y-1}{10}=\frac{z-1}{17}$.

---

22. The equations of motion of a rocket are x = 2t, y = −4t, z = 4t, where t is in seconds and the coordinates are in km. What is the path of the rocket? At what distances will the rocket be from the starting point O(0, 0, 0) and from the line $\vec{R} = 20\hat{i} - 10\hat{j} + 40\hat{k} + \mu(10\hat{i} - 20\hat{j} + 10\hat{k})$ in 10 seconds?

Solution : Eliminating t: $\frac{x}{2}=\frac{y}{-4}=\frac{z}{4}$ — a straight line through O with direction ratios 1, −2, 2.

At t = 10 s the rocket is at P(20, −40, 40):

$OP = \sqrt{400 + 1600 + 1600} = 60$ km

Distance from the line (point A(20, −10, 40), direction $\vec{b}$ = (1, −2, 1)):

$\vec{AP}$ = (0, −30, 0); $\vec{AP} \times \vec{b}$ = (−30, 0, 30), magnitude $30\sqrt{2}$

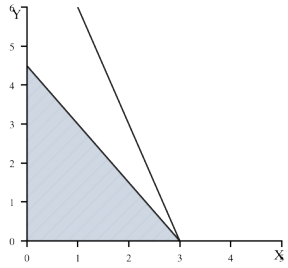
$$d = \frac{30\sqrt{2}}{\sqrt{6}} = 10\sqrt{3} \text{ km}$$

$\therefore$ Path: straight line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$; 60 km from O and $10\sqrt{3}$ km from the given line.

Note: Case-study questions are recommended from this chapter for **CBSE Board 2026-27**.
Practise them here: **Class XII Case Studies (click here)**

Chapter 12 – Linear Programming

1. Solve the following LPP graphically: Maximise P = 70x + 40y subject to 3x + 2y ≤ 9, 3x + y ≤ 9, x ≥ 0, y ≥ 0.



Solution : Draw the lines 3x + 2y = 9 (through (3, 0), (0, 4.5)) and 3x + y = 9 (through (3, 0), (0, 9)). The feasible region is the triangle OAB shown.

Corner points and values of P:

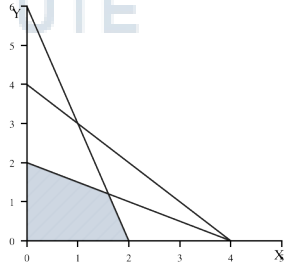
O(0, 0): 0; A(3, 0): 210; B(0, 4.5): 180

The region is bounded, so the maximum occurs at a corner point.

$\therefore$ Maximum P = 210 at x = 3, y = 0.

---

2. Find graphically the maximum value of Z = 2x + 5y subject to 2x + 4y ≤ 8, 3x + y ≤ 6, x + y ≤ 4, x ≥ 0, y ≥ 0.



Solution : Lines: 2x + 4y = 8 [(4, 0), (0, 2)], 3x + y = 6 [(2, 0), (0, 6)], x + y = 4 [(4, 0), (0, 4)]. The constraint x + y ≤ 4 is redundant.

Intersection of 2x + 4y = 8 and 3x + y = 6:

$$x + 2y = 4 \text{ and } 3x + y = 6 \Rightarrow x = \frac{8}{5}, y = \frac{6}{5}$$

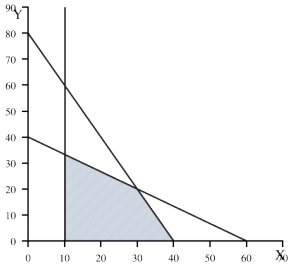
Corner points and values of Z:

$$(0, 0): 0; (2, 0): 4; \left(\frac{8}{5}, \frac{6}{5}\right): \frac{46}{5}; (0, 2): 10$$

$\therefore$ Maximum Z = 10 at (0, 2).

---

3. Solve graphically: Maximise Z = 7x + 10y subject to 4x + 6y ≤ 240, 6x + 3y ≤ 240, x ≥ 10, y ≥ 0.



Solution : Intersection of 4x + 6y = 240 and 6x + 3y = 240:

2x + 3y = 120, 2x + y = 80 ⇒ y = 20, x = 30

Other corners: (10, 0), (40, 0) and (10, $\frac{100}{3}$) where x = 10 meets 4x + 6y = 240.

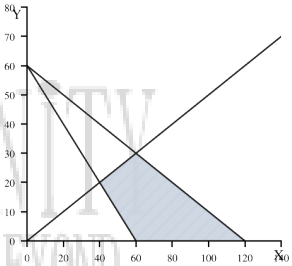
Values of Z:

(10, 0): 70; (40, 0): 280; (30, 20): 410; $\left(10, \frac{100}{3}\right)$: $\frac{1210}{3} \approx 403.3$

∴ Maximum Z = 410 at (30, 20).

---

4. Solve graphically: Minimise Z = 5x + 10y subject to x + 2y ≤ 120, x + y ≥ 60, x − 2y ≥ 0, x, y ≥ 0.



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Solution : Corner points of the feasible region:

x + y = 60 and y = 0: (60, 0); x + 2y = 120 and y = 0: (120, 0)

x + 2y = 120 and x = 2y: (60, 30); x + y = 60 and x = 2y: (40, 20)

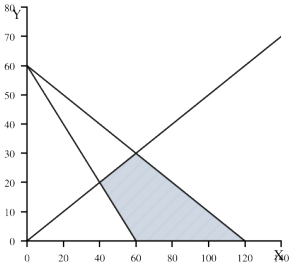
Values of Z:

(60, 0): 300; (120, 0): 600; (60, 30): 600; (40, 20): 400

∴ Minimum Z = 300 at (60, 0).

---

5. Solve the LPP of the previous question for the maximum of Z = 5x + 10y under the same constraints. Comment on the solution.



Solution : From the same table of corner values:

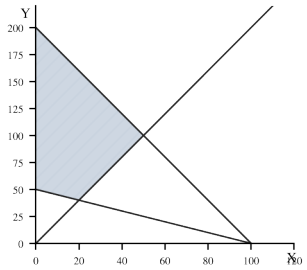
(120, 0): 600 and (60, 30): 600 are both the greatest value

When the maximum occurs at two corner points, it occurs at every point of the segment joining them.

∴ Maximum Z = 600 at every point of the line segment joining (120, 0) and (60, 30).

---

6. Solve graphically: Maximise Z = x + 2y subject to x + 2y ≥ 100, 2x − y ≤ 0, 2x + y ≤ 200, x, y ≥ 0.



Solution : Corner points:

x + 2y = 100 and x = 0: (0, 50); x + 2y = 100 and y = 2x: (20, 40)

y = 2x and 2x + y = 200: (50, 100); 2x + y = 200 and x = 0: (0, 200)

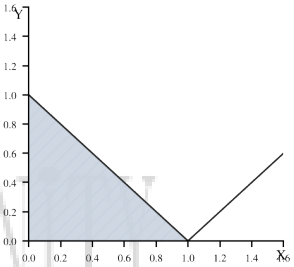
Values of Z:

(0, 50): 100; (20, 40): 100; (50, 100): 250; (0, 200): 400

∴ Maximum Z = 400 at (0, 200).

---

7. Find the maximum of Z = x + y subject to y ≥ |x| − 1, y ≤ 1 − |x|, x ≥ 0, y ≥ 0.



Solution : For x ≥ 0, the constraints become y ≥ x − 1 and y ≤ 1 − x. With y ≥ 0, the feasible region is the triangle with vertices (0, 0), (1, 0), (0, 1).

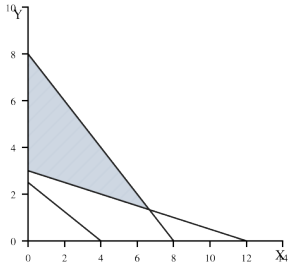
Z(0, 0) = 0, Z(1, 0) = 1, Z(0, 1) = 1

The maximum occurs at two corners.

∴ Maximum Z = 1 at every point of the segment joining (1, 0) and (0, 1).

---

8. Solve graphically: Minimise Z = 30x + 20y subject to x + y ≤ 8, x + 4y ≥ 12, 5x + 8y ≥ 20, x, y ≥ 0.



Solution : The constraint 5x + 8y ≥ 20 is automatically satisfied in the region (redundant). Corner points:

x + 4y = 12 and x = 0: (0, 3); x + y = 8 and x = 0: (0, 8)

x + y = 8 and x + 4y = 12: $3y = 4 \Rightarrow \left(\frac{20}{3}, \frac{4}{3}\right)$

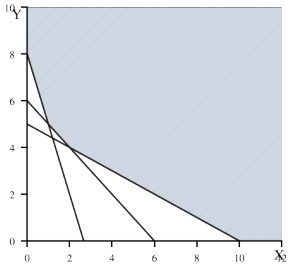
Values of Z:

(0, 3): 60; (0, 8): 160; $\left(\frac{20}{3}, \frac{4}{3}\right)$: $\frac{680}{3}$

∴ Minimum Z = 60 at (0, 3).

---

9. Minimise Z = 16x + 20y subject to x + 2y ≥ 10, x + y ≥ 6, 3x + y ≥ 8, x, y ≥ 0.



Solution : The feasible region is unbounded. Corner points:

(0, 8); 3x + y = 8 and x + y = 6: (1, 5); x + y = 6 and x + 2y = 10: (2, 4); (10, 0)

Values of Z:

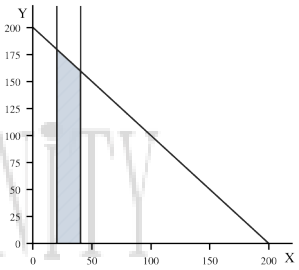
(0, 8): 160; (1, 5): 116; (2, 4): 112; (10, 0): 160

Smallest value is 112. Since the region is unbounded, check the half-plane 16x + 20y < 112: it has no point in common with the feasible region.

∴ Minimum Z = 112 at (2, 4).

---

10. Solve graphically: Maximise Z = 400x + 300y subject to x + y ≤ 200, x ≤ 40, x ≥ 20, y ≥ 0.



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Solution : Corner points of the feasible region:

(20, 0), (40, 0), (40, 160), (20, 180)

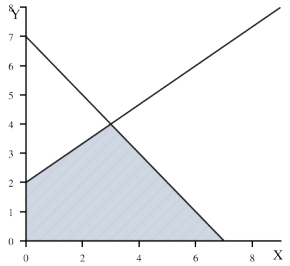
Values of Z:

(20, 0): 8000; (40, 0): 16000; (40, 160): 64000; (20, 180): 62000

∴ Maximum Z = 64000 at (40, 160).

---

11. Minimise Z = 13x − 15y subject to x + y ≤ 7, 2x − 3y + 6 ≥ 0, x ≥ 0, y ≥ 0.



Solution : Corner points:

(0, 0); (7, 0); (0, 2) where 2x − 3y + 6 = 0 meets x = 0

x + y = 7 and 2x − 3y = −6: 5x = 15 ⇒ (3, 4)

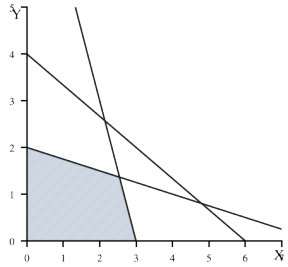
Values of Z:

(0, 0): 0; (7, 0): 91; (3, 4): −21; (0, 2): −30

∴ Minimum Z = −30 at (0, 2).

---

12. Maximise Z = x + y subject to x + 4y ≤ 8, 2x + 3y ≤ 12, 3x + y ≤ 9, x ≥ 0, y ≥ 0.



Solution : The constraint 2x + 3y ≤ 12 is redundant. Corner points:

(0, 0), (3, 0), (0, 2)

x + 4y = 8 and 3x + y = 9: x = 8 − 4y ⇒ 24 − 11y = 9 ⇒ $\left(\frac{28}{11}, \frac{15}{11}\right)$

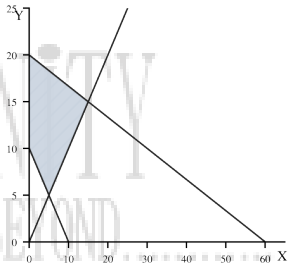
Values of Z:

(0, 0): 0; (3, 0): 3; (0, 2): 2; $\left(\frac{28}{11}, \frac{15}{11}\right)$: $\frac{43}{11}$

∴ Maximum Z = $\frac{43}{11}$ at $\left(\frac{28}{11}, \frac{15}{11}\right)$.

---

13. Determine graphically the minimum and maximum values of Z = 3x + 9y subject to x + 3y ≤ 60, x + y ≥ 10, x ≤ y, x ≥ 0, y ≥ 0.



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Solution : Corner points:

(0, 10), (5, 5), (15, 15), (0, 20)

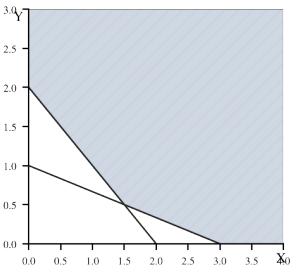
Values of Z:

(0, 10): 90; (5, 5): 60; (15, 15): 180; (0, 20): 180

∴ Minimum Z = 60 at (5, 5); maximum Z = 180 at every point of the segment joining (15, 15) and (0, 20).

---

14. Minimise Z = 3x + 5y subject to x + 3y ≥ 3, x + y ≥ 2, x, y ≥ 0.



Solution : The feasible region is unbounded. Corner points:

(3, 0); x + 3y = 3 and x + y = 2: $\left(\frac{3}{2}, \frac{1}{2}\right)$; (0, 2)

Values of Z:

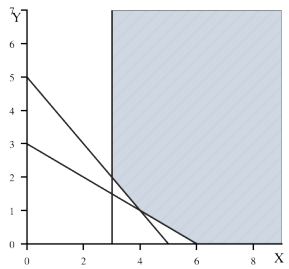
(3, 0): 9; $\left(\frac{3}{2}, \frac{1}{2}\right)$: 7; (0, 2): 10

The half-plane $3x + 5y < 7$ has no point in common with the feasible region.

$\therefore$ Minimum Z = 7 at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

---

15. Maximise $Z = -x + 2y$ subject to $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$. Comment on the result.



Solution : The feasible region is unbounded. Corner points:

(6, 0); $x + 2y = 6$ and $x + y = 5$: (4, 1); $x = 3$ and $x + y = 5$: (3, 2)

Values of Z:

(6, 0): −6; (4, 1): −2; (3, 2): 1

Largest corner value is 1, but the open half-plane $-x + 2y > 1$ has points in common with the region (for example (3, 5) gives Z = 7).

$\therefore$ Z has no maximum value (the region is unbounded in the direction of increasing Z).

Chapter 13 – Probability

1. Assume that each child born is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls, given that (i) the youngest is a girl (ii) at least one is a girl?

Solution : Sample space (elder, younger): S = {BB, BG, GB, GG}, each with probability $\frac{1}{4}$. Let A = both girls = {GG}.

(i) B = youngest is a girl = {BG, GG}:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) C = at least one girl = {BG, GB, GG}:

$$P(A \mid C) = \frac{1/4}{3/4} = \frac{1}{3}$$

$\therefore$ (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$.

---

2. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution : Let F = red die shows 1, 2 or 3: $3 \times 6 = 18$ outcomes, $P(F) = \frac{18}{36}$.

$E \cap F$ = sum 8 with red die < 4: (red 2, black 6), (red 3, black 5) — 2 outcomes.

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{2/36}{18/36}$$

$\therefore P(E \mid F) = \frac{1}{9}$.

---

3. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event 'number obtained is even' and B be the event 'number obtained is red'. Find whether A and B are independent.

Solution :

$A = \{2, 4, 6\}$, $P(A) = \frac{1}{2}$; $B = \{1, 2, 3\}$, $P(B) = \frac{1}{2}$

$A \cap B = \{2\}$, $P(A \cap B) = \frac{1}{6}$

$P(A) \cdot P(B) = \frac{1}{4} \neq \frac{1}{6}$

$\therefore$ A and B are not independent.

---

4. If A and B are independent events such that P(A) = 0.3 and P(A ∪ B) = 0.5, find P(A | B) − P(B | A).

Solution : For independent events $P(A \cap B) = P(A)P(B)$:

$$0.5 = 0.3 + P(B) - 0.3P(B) \Rightarrow 0.7P(B) = 0.2 \Rightarrow P(B) = \frac{2}{7}$$

Independence also gives $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$:

$$P(A \mid B) - P(B \mid A) = 0.3 - \frac{2}{7} = \frac{21 - 20}{70}$$

$\therefore$ Required value $= \frac{1}{70}$.

---

5. If $P(F) = \frac{1}{5}$ and $P(E) = \frac{1}{2}$, and E and F are independent events, find P(not (E ∪ F)).

Solution : For independent events, the complements are also independent:

$$P(E' \cap F') = P(E')P(F') = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$$

By De Morgan's law, $(E \cup F)' = E' \cap F'$:

$\therefore P((E \cup F)') = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}.$

---

6. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits it is $\frac{2}{5}$. If both try independently, find the probability that the target is hit.

Solution : The target is missed only if both miss:

$$P(\text{both miss}) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$$

$\therefore P(\text{target is hit}) = 1 - \frac{2}{5} = \frac{3}{5}.$

---

7. The probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try independently, find the probability that (i) the problem is solved (ii) exactly one of them solves it.

Solution : (i) P(solved) = 1 − P(neither solves):

$$1 - \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}$$

(ii) Exactly one solves:

$$\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$\therefore$ (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$.

---

8. A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Find the probability that the problem is solved.

Solution : The problem remains unsolved only if all three fail (independently):

$$P(\text{none solves}) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

$\therefore P(\text{problem is solved}) = 1 - \frac{1}{4} = \frac{3}{4}.$

---

9. A speaks the truth in 75% of the cases while B does so in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact? Is the statement of B always true?

Solution : They contradict when exactly one of them speaks the truth:

$P = P(\text{A true})P(\text{B false}) + P(\text{A false})P(\text{B true}) = 0.75 \times 0.10 + 0.25 \times 0.90$

$= 0.075 + 0.225 = 0.30$

B speaks the truth only in 90% of cases, so his statement is not always true.

$\therefore$ They contradict each other in 30% of the cases; no, B's statement is not always true.

---

10. A bag contains 3 red and 7 black balls. Two balls are selected at random one by one without replacement. If the second ball is red, what is the probability that the first ball is also red?

Solution : Let R_1, R_2 be 'first red' and 'second red'.

$$P(R_1 \cap R_2) = \frac{3}{10} \cdot \frac{2}{9} = \frac{6}{90}$$

$$P(R_2) = \frac{3}{10} \cdot \frac{2}{9} + \frac{7}{10} \cdot \frac{3}{9} = \frac{6+21}{90} = \frac{27}{90}$$

$$P(R_1 \mid R_2) = \frac{6/90}{27/90}$$

$\therefore$ Required probability $= \frac{2}{9}$.

---

11. A purse contains 3 silver and 6 copper coins and a second purse contains 4 silver and 3 copper coins. If a coin is drawn at random from one of the two purses, find the probability that it is a silver coin.

Solution : Each purse is chosen with probability $\frac{1}{2}$. By the theorem of total probability:

$$P(\text{silver}) = \frac{1}{2} \cdot \frac{3}{9} + \frac{1}{2} \cdot \frac{4}{7} = \frac{1}{6} + \frac{2}{7}$$

$\therefore P(\text{silver}) = \frac{19}{42}$.

---

12. There are two bags. Bag I contains 1 red and 3 white balls and Bag II contains 3 red and 5 white balls. A bag is selected at random and a ball is drawn from it. Find the probability that the ball drawn is red.

Solution : By the theorem of total probability:

$$P(\text{red}) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{8} = \frac{1}{8} + \frac{3}{16}$$

$\therefore P(\text{red}) = \frac{5}{16}$.

---

13. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears, bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and the other black.

Solution :

$$P(A) = \frac{2}{6} = \frac{1}{3},\ P(B) = \frac{2}{3}$$

Probability of one red and one black:

$$\text{From A: } \frac{4 \times 6}{{}^{10}C_2} = \frac{24}{45};\ \text{from B: } \frac{7 \times 3}{45} = \frac{21}{45}$$

Total probability:

$$\frac{1}{3} \cdot \frac{24}{45} + \frac{2}{3} \cdot \frac{21}{45} = \frac{24 + 42}{135} = \frac{66}{135}$$

$\therefore$ Required probability $= \frac{22}{45}$.

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14. Of the students in a college, 60% reside in the hostel and 40% are day scholars. 30% of hostellers and 20% of day scholars attain A grade. A student chosen at random has an A grade. What is the probability that the student is a hosteller?

Solution : Let H = hosteller, D = day scholar, A = A grade.

$P(H) = 0.6,\ P(D) = 0.4,\ P(A \mid H) = 0.3,\ P(A \mid D) = 0.2$

By Bayes' theorem:

$$P(H \mid A) = \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.4 \times 0.2} = \frac{0.18}{0.26}$$

$\therefore P(H \mid A) = \frac{9}{13}$.

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15. Of the students in a school, 30% have 100% attendance and 70% are irregular. 70% of students with 100% attendance and 10% of irregular students attain A grade. A student chosen at random has an A grade. What is the probability that the student has 100% attendance?

Solution : Let E_1 = 100% attendance, E_2 = irregular, A = A grade.

$P(E_1) = 0.3,\ P(E_2) = 0.7,\ P(A \mid E_1) = 0.7,\ P(A \mid E_2) = 0.1$

By Bayes' theorem:

$$P(E_1 \mid A) = \frac{0.3 \times 0.7}{0.3 \times 0.7 + 0.7 \times 0.1} = \frac{0.21}{0.28}$$

$\therefore P(E_1 \mid A) = \frac{3}{4}$.

---

16. In a factory which manufactures bolts, machines A, B and C manufacture 30%, 50% and 20% of the bolts, and 3%, 4% and 1% of their outputs respectively are defective. A bolt is drawn at random and is found to be defective. Find the probability that it is not manufactured by machine B.

Solution :

$P(D) = 0.3 \times 0.03 + 0.5 \times 0.04 + 0.2 \times 0.01 = 0.009 + 0.020 + 0.002 = 0.031$

By Bayes' theorem:

$$P(B \mid D) = \frac{0.020}{0.031} = \frac{20}{31}$$

$\therefore P(\text{not } B \mid D) = 1 - \frac{20}{31} = \frac{11}{31}.$

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17. Bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B.

Solution :

$$P(A) = P(B) = \frac{1}{2};\ P(R \mid A) = \frac{3}{5},\ P(R \mid B) = \frac{5}{9}$$

By Bayes' theorem:

$$P(B \mid R) = \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{5}{9}} = \frac{5/9}{52/45}$$

$\therefore P(B \mid R) = \frac{25}{52}.$

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18. Bag A contains 5 black and 3 red balls while bag B contains 4 black and 4 red balls. Two balls are transferred at random from bag A to bag B and then a ball is drawn from bag B. If this ball is red, find the probability that two red balls were transferred.

Solution : Cases for the transfer (out of ${}^8C_2 = 28$):

E_1: 2 red, $P = \frac{3}{28}$; E_2: 2 black, $P = \frac{10}{28}$; E_3: 1 red 1 black, $P = \frac{15}{28}$

Bag B then has 10 balls; P(red drawn):

$$P(R \mid E_1) = \frac{6}{10},\ P(R \mid E_2) = \frac{4}{10},\ P(R \mid E_3) = \frac{5}{10}$$

By Bayes' theorem:

$$P(E_1 \mid R) = \frac{3 \times 6}{3 \times 6 + 10 \times 4 + 15 \times 5} = \frac{18}{133}$$

$\therefore$ Required probability $= \frac{18}{133}.$

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19. A man is known to speak the truth 3 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution : Let S = six occurs, T = he reports a six.

$$P(S) = \frac{1}{6},\ P(S') = \frac{5}{6};\ P(T \mid S) = \frac{3}{5},\ P(T \mid S') = \frac{2}{5}$$

By Bayes' theorem:

$$P(S \mid T) = \frac{\frac{1}{6} \cdot \frac{3}{5}}{\frac{1}{6} \cdot \frac{3}{5} + \frac{5}{6} \cdot \frac{2}{5}} = \frac{3}{3 + 10}$$

$$\therefore P(S \mid T) = \frac{3}{13}.$$

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20. In answering a question on a multiple-choice test, a student either knows the answer or guesses. The probability that he knows the answer is $\frac{3}{4}$. If he guesses, the probability of a correct answer is $\frac{1}{4}$. What is the probability that he knew the answer, given that he answered it correctly?

Solution : Let K = knows, G = guesses, C = correct.

$$P(K) = \frac{3}{4},\ P(G) = \frac{1}{4};\ P(C \mid K) = 1,\ P(C \mid G) = \frac{1}{4}$$

By Bayes' theorem:

$$P(K \mid C) = \frac{\frac{3}{4} \cdot 1}{\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{12/16}{13/16}$$

$$\therefore P(K \mid C) = \frac{12}{13}.$$

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21. A and B throw a pair of dice alternately till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning if A starts first.

Solution :

$$P(\text{total } 10) = \frac{3}{36} = \frac{1}{12} = p;\ q = \frac{11}{12}$$

A wins on the 1st, 3rd, 5th, … throws:

$$P(A) = p + q^2p + q^4p + \ldots = \frac{p}{1 - q^2} = \frac{1/12}{1 - 121/144} = \frac{1}{12} \cdot \frac{144}{23} = \frac{12}{23}$$

$$P(B) = 1 - \frac{12}{23}$$

$$\therefore P(\text{A wins}) = \frac{12}{23},\ P(\text{B wins}) = \frac{11}{23}.$$

22. A and B throw a pair of dice alternately. A wins if he gets a total of 7 and B wins if he gets a total of 10. If A starts the game, find the probability that B wins.

Solution :

$$P(\text{A gets 7}) = \frac{6}{36} = \frac{1}{6};\ P(\text{B gets 10}) = \frac{3}{36} = \frac{1}{12}$$

B wins on his 1st turn, or after one more full round, and so on:

$$P(B) = \frac{5}{6}\cdot\frac{1}{12} + \left(\frac{5}{6}\cdot\frac{11}{12}\right)\frac{5}{6}\cdot\frac{1}{12} + \ldots = \frac{\frac{5}{72}}{1-\frac{55}{72}}$$

$\therefore P(\text{B wins}) = \frac{5}{17}$.

23. Let X denote the number of hours you study during a randomly selected school day. P(X = x) = 0.1 if x = 0; kx if x = 1 or 2; k(5 − x) if x = 3 or 4; 0 otherwise. (a) Find k. (b) Find the probability that you study (i) at least two hours (ii) exactly two hours (iii) at most two hours.

Solution : (a) Sum of probabilities = 1:

$0.1 + k + 2k + 2k + k = 1 \Rightarrow 6k = 0.9 \Rightarrow k = 0.15$

(b)

(i) $P(X \geq 2) = 2k + 2k + k = 5k = 0.75$

(ii) $P(X = 2) = 2k = 0.30$

(iii) $P(X \leq 2) = 0.1 + k + 2k = 0.55$

$\therefore k = 0.15$; (i) 0.75 (ii) 0.3 (iii) 0.55.

24. Out of a group of 50 people, 20 always speak the truth. Two persons are selected at random (without replacement). Find the probability distribution of the number of selected persons who always speak the truth.

Solution : Let X = number of truthful persons selected; X = 0, 1, 2. Total ways = $^{50}C_2 = 1225$.

$$P(X=0) = \frac{^{30}C_2}{1225} = \frac{435}{1225} = \frac{87}{245}$$

$$P(X=1) = \frac{20\times 30}{1225} = \frac{600}{1225} = \frac{24}{49}$$

$$P(X=2) = \frac{^{20}C_2}{1225} = \frac{190}{1225} = \frac{38}{245}$$

$\therefore$ X: 0, 1, 2 with probabilities $\frac{87}{245}, \frac{24}{49}, \frac{38}{245}$.

25. A pair of dice is thrown simultaneously. If X denotes the absolute difference of the numbers obtained, find the probability distribution of X and its mean.

Solution : Count outcomes for each value of |a − b| (out of 36):

X = 0: 6; X = 1: 10; X = 2: 8; X = 3: 6; X = 4: 4; X = 5: 2

Probabilities: $\frac{6}{36}, \frac{10}{36}, \frac{8}{36}, \frac{6}{36}, \frac{4}{36}, \frac{2}{36}$

Mean:

$$E(X) = \frac{0+10+16+18+16+10}{36} = \frac{70}{36}$$

$\therefore$ Distribution as above; mean $= \frac{35}{18}$.

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26. Two numbers are selected at random (without replacement) from the first six positive even integers. If X denotes the larger of the two numbers, find the probability distribution of X and its mean.

Solution : The numbers are 2, 4, 6, 8, 10, 12; number of pairs = ${}^6C_2 = 15$.

X = larger number; X = k is obtained by pairing k with each smaller number:

X = 4: 1 pair; 6: 2; 8: 3; 10: 4; 12: 5

Probabilities: $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}$

$$E(X) = \frac{4+12+24+40+60}{15} = \frac{140}{15}$$

$\therefore$ Mean $= \frac{28}{3}$.

Note: Case-study questions are recommended from this chapter for **CBSE Board 2026-27**.
Practise them here: **Class XII Case Studies (click here)**

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